

ON A POISSON SPACE OF BILINEAR FORMS WITH A POISSON LIE ACTION

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ABSTRACT. We consider the space \mathcal{A} of bilinear forms on \mathbb{C}^N with defining matrix \mathbb{A} endowed with the quadratic Poisson structure studied by the authors in [3]. We classify all possible quadratic brackets on $(B, \mathbb{A}) \in GL_N \times \mathcal{A}$ with the property that the natural action $\mathbb{A} \mapsto B\mathbb{A}B^T$ of the GL_N Poisson–Lie group on the space \mathcal{A} is a Poisson action thus endowing \mathcal{A} with the structure of Poisson space. Beside the product Poisson structure on $GL_N \times \mathcal{A}$ we find two more (dual to each other) structures for which (in contrast to the product Poisson structure) we can implement the reduction to the space of bilinear forms with block upper triangular defining matrices by Dirac procedure. We consider the generalisation of the above construction to triples $(B, C, \mathbb{A}) \in GL_N \times GL_N \times \mathcal{A}$ with the Poisson action $\mathbb{A} \mapsto B\mathbb{A}C^T$ and show that \mathcal{A} then acquires the structure of Poisson symmetric space. We study also the generalisation to chains of transformations and to the quantum and quantum affine algebras and the relation between the construction of Poisson symmetric spaces and that of the Poisson groupoid.

1. INTRODUCTION

In this paper, we identify bilinear forms on \mathbb{C}^N

$$\langle x, y \rangle := x^T \mathbb{A} y, \quad \forall x, y \in \mathbb{C}^N, \quad \mathbb{A} \in \text{Mat}_N(\mathbb{C}),$$

with their defining matrix \mathbb{A} . We denote by \mathcal{A} the space of such bilinear forms.

In [3], the authors studied a quadratic Poisson algebra structure on the space \mathcal{A} of bilinear forms on \mathbb{C}^N with the property that for any $n, m \in \mathbb{N}$ such that $nm = N$, the restriction of the Poisson algebra to the space $\mathcal{A}_{n,m}$ of bilinear forms with block-upper-triangular (b.u.t.) defining matrix composed from blocks of size $m \times m$ is Poisson:

$$(1.1) \quad \{a_{i,j}, a_{k,l}\} = (\text{sign}(j-l) + \text{sign}(i-k))a_{i,l}a_{k,j} + (\text{sign}(j-k) + 1)a_{j,l}a_{i,k} + (\text{sign}(i-l) - 1)a_{l,j}a_{k,i}.$$

These algebras were studied previously in the upper-triangular case in [10], [20], [21] and in the case of 2×2 blocks in [18], [19] (see also monograph [17]) in relation to various algebraic and geometric systems.

In the r -matrix notation explained in Appendix A the same bracket (1.1) can be written in a more concise form:

$$(1.2) \quad \{\overset{1}{\mathbb{A}} \otimes \overset{2}{\mathbb{A}}\} = r_{12}(\overset{1}{\mathbb{A}} \otimes \overset{2}{\mathbb{A}}) - (\overset{1}{\mathbb{A}} \otimes \overset{2}{\mathbb{A}})r_{12} + \overset{1}{\mathbb{A}}r_{12}^{t_1}\overset{2}{\mathbb{A}} - \overset{2}{\mathbb{A}}r_{12}^{t_1}\overset{1}{\mathbb{A}},$$

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where

$$(1.3) \quad r_{12} = \sum_i \overset{1}{E}_{i,i} \otimes \overset{2}{E}_{i,i} + 2 \sum_{i>j} \overset{1}{E}_{i,j} \otimes \overset{2}{E}_{j,i} = 2 \sum_{i,j} \theta(i-j) \overset{1}{E}_{i,j} \otimes \overset{2}{E}_{j,i},$$

with $\theta(x) = \{1, x > 0, 1/2, x = 0, 0, x < 0\}$ is the classical (trigonometrical) r -matrix.

It is natural to consider the following action of GL_N on the space of bilinear forms \mathcal{A} :

$$(1.4) \quad \forall B \in GL_N, \quad B : \mathbb{A} \mapsto \mathbb{A}' := BAB^T.$$

In [3], the groupoid Γ of morphisms of the space $\mathcal{A}_{n,m}$ of b.u.t. bilinear forms was defined in such a way that all morphisms automatically preserve the Poisson algebra on $\mathcal{A}_{n,m}$. The condition for transformation (1.4) to be Poisson is in a sense opposite to the standard construction of a Poisson (symplectic) groupoid [12], [23] in which the source $s : (\mathbb{A}, B) \rightarrow \mathbb{A}$ and target $t : (\mathbb{A}, B) \rightarrow BAB^T$ projections are respectively anti-Poisson and Poisson. We present our treatment of a (possibly more familiar to the reader) Poisson groupoid construction in Sec. 10.

In the main part of this paper we show that it is in fact more natural to ask the following question along the Jiang-Hua Lu approach [14]: considering the action of the GL_N Poisson Lie group on \mathcal{A} we must classify all possible Poisson brackets on $GL_N \times \mathcal{A}$ such that this action is Poisson and \mathcal{A} is a Poisson space in the sense of [7].

We assume that the bracket between B -matrices has the standard Lie–Poisson form

$$(1.5) \quad \{ \overset{1}{B} \otimes \overset{2}{B} \} = r_{12} (\overset{1}{B} \otimes \overset{2}{B}) - (\overset{1}{B} \otimes \overset{2}{B}) r_{12},$$

and prove (see Theorem 2.1) that in order for the action of GL_N on \mathcal{A} to be Poisson, the brackets between entries of the \mathbb{A} -matrix and entries of the B -matrix must have a special quadratic form

$$(1.6) \quad \{ \overset{1}{B} \otimes \overset{2}{\mathbb{A}} \} = \overset{1}{B} Q_{12} \overset{2}{\mathbb{A}} + \overset{1}{B} \overset{2}{\mathbb{A}} Q_{12}^{t_2},$$

with the matrix Q_{12} taking only three possible choices:

- (i) $Q_{12} = 0$;
- (ii) $Q_{12} = -r_{12}^{t_2}$;
- (iii) $Q_{12} = r_{12}^{t_1}$.

In the text, we refer to these choices by their numbers (i), (ii), and (iii). The Poisson structure (i) on $GL_N \times \mathcal{A}$ is the usual product Poisson structure (i.e., with $Q_{12} = 0$), while the other two are dual to each other (see Lemma 2.3).

Stress that in none of the above cases $GL_N \times \mathcal{A} \rightrightarrows \mathcal{A}$ is a Poisson groupoid (in particular the target map $\beta : (B, \mathbb{A}) \rightarrow BAB^T$ is not an anti-Poisson map). We can nevertheless endow $GL_N \times GL_N$ with a Poisson symmetric Lie group structure and show that \mathcal{A} is the symmetric space associated to it in the sense defined by Fernandes in [8] (see Theorem 3.3 here below). Loosely speaking this means that we can think of \mathcal{A} as the set of matrices \mathbb{A} of the form $\mathbb{A} = BC^T$, where $(B, C) \in GL_N \times GL_N$, and we prove that the Poisson bracket (1.2) between entries of \mathbb{A} is in fact induced by the Poisson structure on $GL_N \times GL_N$ by the identification $\mathbb{A} = BC^T$.

Again, we classify all quadratic Poisson brackets on $GL_N \times GL_N \times \mathcal{A}$ for which the Poisson Lie group action on \mathcal{A} defined by

$$(\mathbb{A}, B, C) \mapsto B\mathbb{A}C^T, \quad \forall (B, C) \in GL_N \times GL_N$$

is Poisson (see Lemma 3.1).

This raises the following question: if we identify $\mathbb{A} = BC^T$, is it true that the Poisson bracket on $GL_N \times GL_N \times \mathcal{A}$ is induced from the bracket among entries of B and C ? It turns out that the correct way to formulate (and indeed answer) this question is in terms of *chains*: we introduce Poisson brackets on $\bigotimes_1^K (GL_N \times GL_N) \otimes \mathcal{A}$ such that the action of the Poisson Lie group $\bigotimes_K (GL_N \times GL_N)$ on \mathcal{A} is Poisson for every K . Then, if we identify $\mathbb{A} = B_1 C_1^T$, where (B_1, C_1) is an element in the first copy of $GL_N \times GL_N$, the Poisson brackets on $\bigotimes_2^K (GL_N \times GL_N) \otimes \mathcal{A}$ are induced by the one on $\bigotimes_1^K (GL_N \times GL_N) \otimes \mathcal{A}$ (see Section 5).

In Section 6 we classify all central elements for all Poisson brackets on $GL_N \times \mathcal{A}$, $GL_N \times GL_N \times \mathcal{A}$, and on chains of B -matrices and (B, C) -pairs.

The next set of results deals with the natural question of reductions to the space $\mathcal{A}_{n,m}$ of b.u.t. bilinear forms. In order to implement this reduction, we need to introduce the set of constraints (7.2) and (7.3) on blocks $A_{I,J}$ and $A'_{I,J}$ of the respective matrices \mathbb{A} and $\mathbb{A}' := BAB^T$.

This is where the non trivial Poisson brackets (ii) or (iii) between A and B become necessary. In fact, these constraints do not Poisson commute with all other elements on the constraint surface, therefore this reduction is not Poisson and a Dirac reduction is needed. However in order for the Dirac reduction to work, the matrix given by all Poisson brackets between constraints must be non-degenerate on the constraint surface, and this is only possible when A and B do not Poisson commute with each other.

We illustrate in detail how the Dirac procedure works in the case of upper-triangular matrices \mathbb{A} , i.e. on the space $\mathcal{A}_{n,1}$.

It is interesting to observe that in this case we can solve the constraint equations for \mathbb{A} thus obtaining the entries $a_{i,j}$ as functions $F_{i,j}[B]$. In this way, we identify \mathbb{A} with the new upper-triangular matrix \mathbb{F} . In what follows, we can proceed in two, very different, ways. In our original formulation of the Poisson space, we can treat equations $\mathbb{A}_{i,j} - F_{i,j}[B] = 0$, $i < j$, $\mathbb{A}_{i,j} - \delta_{i,j} = 0$, $i \geq j$, as an equivalent set of second-kind constraints implementing the same Dirac procedure as above (Sec. 7). On the other hand, we can just induce the Poisson brackets on \mathbb{A} and between \mathbb{A} and B from the Lie Poisson brackets on B , which, as we demonstrate in Sec. 10 results in the structure of Poisson (symplectic) groupoid on the pair (\mathbb{A}, B) .

In Secs. 8 and 9, we use the r -matrix formalism to quantise all brackets and produce a quantum affine version of the Poisson algebra (ii) on $GL_N \times \mathcal{A}$ and prove the quantum Jacobi relations (the formulae for the Poisson algebra (iii) can be deduced by duality).

We finally address the symplectic groupoid construction by [12], [23], [1]. Assuming that the brackets on $F_{i,j}[B]$ are induced by those on the entries of B we obtain that $F_{i,j}[B]$ satisfy the same relations as the entries of \mathbb{A} with opposite sign. The thus found brackets on the set of (\mathbb{F}, B) pairs are again quadratic and admit an r -matrix form of writing. We can therefore extend these brackets to the general case $(\mathbb{F}, B) \in GL_N \times GL_N$. We find that the mapping $\mathbb{F} \mapsto \tilde{\mathbb{F}} := B\mathbb{F}B^T$ is then indeed an *antiautomorphism* of the Poisson algebra for \mathbb{F} whereas all entries

of \mathbb{F} and $B\mathbb{F}B^T$ mutually Poisson commute. This is in accordance with the factorization property of the symplectic groupoid [23], [13]. We then show that this alternative bracket admits upper-triangular and block-upper-triangular Poisson reductions without involving the Dirac procedure and looks therefore quite attractive on the first sight. Its disadvantage, to our opinion, is that it does not produce a nontrivial dynamics resulting just in the appearance of two separate copies of the original Poisson algebra for \mathbb{A} (with opposite signs) sharing the same central elements generated by $\det(\mathbb{F} + \lambda\mathbb{F}^T) = \det(\tilde{\mathbb{F}} + \lambda\tilde{\mathbb{F}}^T)$. It nevertheless satisfy the definition of the Poisson (symplectic) groupoid [16].

Remark 1.1. It is worth noting that \mathcal{A} arises naturally as the unipotent radical of Borel subgroups of complex simple Lie groups, can be identified with Schubert cells on flag varieties. In this way the setting of our paper can be related to Goodearl and Yakimov work [11]. The investigation on the exact relationship between the Poisson structures in that paper and ours is postponed to subsequent publication.¹

2. THE POISSON ALGEBRA ON $GL_N \times \mathcal{A}$

In this section our aim is to find a Poisson structure on $GL_N \times \mathcal{A}$ such that the GL_N -action

$$(2.1) \quad \mathbb{A} \mapsto \mathbb{A}' := B\mathbb{A}B^T,$$

on the space \mathcal{A} is Poisson.

We assume that the bracket (1.2) (or (1.1) in the coordinate form of writing) holds on the space of \mathbb{A} -matrices and look for such brackets $\{\overset{1}{\mathbb{A}} \otimes \overset{2}{B}\}$ and $\{\overset{1}{B} \otimes \overset{2}{B}\}$ that

$$(2.2) \quad \begin{aligned} \{(\overset{1}{B}\overset{1}{A}B^T) \otimes (\overset{2}{B}\overset{2}{A}B^T)\} &= r_{12}(\overset{1}{B}\overset{1}{A}B^T) \otimes (\overset{2}{B}\overset{2}{A}B^T) - (\overset{1}{B}\overset{1}{A}B^T) \otimes (\overset{2}{B}\overset{2}{A}B^T)r_{12} + \\ &+ (\overset{1}{B}\overset{1}{A}B^T)r_{12}^{t_1}(\overset{2}{B}\overset{2}{A}B^T) - (\overset{2}{B}\overset{2}{A}B^T)r_{12}^{t_1}(\overset{1}{B}\overset{1}{A}B^T). \end{aligned}$$

We naturally assume that all the brackets are quadratic and preserve the number of a and b items. We also assume that the bracket between B -matrices has the standard Lie–Poisson form (1.5), or in coordinates:

$$(2.3) \quad \{b_{i,j}, b_{k,l}\} = (\text{sign}(j-l) + \text{sign}(i-k))b_{i,l}b_{k,j}.$$

Theorem 2.1. *Given the Poisson brackets (1.2) between entries of the \mathbb{A} -matrix and (2.3) between entries of the B -matrix, we have exactly three choices for the quadratic brackets between \mathbb{A} and B such that (a) the mapping $\mathbb{A} \mapsto B\mathbb{A}B^T$ is an automorphism of the Poisson algebra (1.2) and (b) the bracket satisfies all the Jacobi relations: all these brackets have the form (1.6) with*

- (i) $Q_{12} = 0$;
- (ii) $Q_{12} = -r_{12}^{t_2}$;
- (iii) $Q_{12} = r_{12}^{t_1}$.

Proof. We begin with the observation that if we begin with evaluating the brackets between entries of the matrices B in (2.2) then, among eight terms four will produce

¹We thank the anonymous referee for this observation

the right-hand side of this relation whether all the remaining terms have to have the structure

$${}^1_2 B B \{ \dots \} {}^1_2 B^T B^T,$$

where the ellipses stand for a combination of r -matrices and elements of matrices \mathbb{A} . The result of evaluation of brackets between entries of matrices \mathbb{A} in (2.2)

has the same form: we obtain expressions sandwiched between ${}^1_2 B B$ and ${}^1_2 B^T B^T$. We therefore assume that, in order to be able to attain proper cancellations, the inter-brackets between B and \mathbb{A} in the expression (2.2) must result in the same sandwiched structure. We therefore look for brackets having a (general) quadratic form

$$(2.4) \quad \{ {}^1_2 B \otimes {}^2_2 \mathbb{A} \} = {}^1_2 B Q_{12} {}^2_2 \mathbb{A} + {}^1_2 B \mathbb{A} R_{12}.$$

The direct calculation then shows that in order to yield an automorphism of the Poisson algebra (1.2) the matrices Q_{12} and R_{12} must satisfy the conditions

$$R_{12} = Q_{21}^T \quad \text{and} \quad Q_{12} - Q_{21} = \kappa P_{12},$$

where $P_{12} = \sum_{i,j} E_{i,j} \otimes E_{j,i}$ is the standard permutation matrix and κ is a constant.

Substituting this ansatz and verifying all the Jacobi relations in the triples $({}^1_2 B, {}^2_2 B, {}^3_3 \mathbb{A})$ and $({}^1_2 B, {}^2_2 \mathbb{A}, {}^3_3 \mathbb{A})$ we observe that, first, $\kappa = 0$ and therefore $Q_{12} = Q_{21}$, and, second, that Q_{12} may assume only the above three forms provided the (B, B) -brackets and the (\mathbb{A}, \mathbb{A}) -brackets are given by the respective formulas (1.2) and (1.5). \square

Although the brackets (ii) and (iii) look less natural than (i), they manifest interesting symmetries as shown by the following result:

Proposition 2.2. *Assuming that $\mathbb{A} \in GL_N(\mathbb{C})$ and $B \in GL_N(\mathbb{C})$, we obtain that in the cases (ii) and (iii) of brackets (1.6), the quantities*

$$(2.5) \quad \mathfrak{A} := B \mathbb{A}^{-T} B^T$$

satisfy the same Poisson algebra (1.2) as both \mathbb{A} and $B \mathbb{A} B^T$.

Proof. Straightforward calculation using the r -matrix form of writing for the corresponding brackets. \square

In order to save the space, we are mostly dealing with the bracket (ii) in what follows; the case of the bracket (iii) is in fact dual to it as proved in the following:

Lemma 2.3. *For \mathbb{A} and B from $GL_N(\mathbb{C})$ we have the following (anti)homomorphism between the (\mathbb{A}, B) -algebras (ii) and (iii). For the quantities $\mathbb{A}' := \mathbb{A}^{-1}$ and $B' = B^{-T}$ the brackets are as follows: the bracket $\{ {}^1_2 \mathbb{A}' \otimes {}^2_2 \mathbb{A}' \}$ has form (1.2) with the overall minus sign, the bracket $\{ {}^1_2 B' \otimes {}^2_2 B' \}$ has form (1.5) with the overall minus sign whereas the bracket $\{ {}^1_2 B' \otimes {}^2_2 \mathbb{A}' \}$ has form (1.6) with the minus sign and with the matrix Q'_{12} of type (iii) if the matrix Q_{12} was of type (ii) and vice versa. (Of course, if \mathbb{A} and B Poisson commute then do the matrices \mathbb{A}' and B' .)*

Proof. Straightforward calculation using the r -matrix form of writing for the corresponding brackets. \square

3. THE POISSON ALGEBRA ON $GL_N \times GL_N \times \mathcal{A}$

We now consider the general transformation

$$(3.1) \quad \mathbb{A} \mapsto B\mathbb{A}C^T,$$

for which we find the brackets between the matrices B and C that preserve the Poisson relations (1.2).

We endow the product $GL_N \times GL_N$ with the following brackets: $\forall(B, C) \in GL_N \times GL_N$

$$(3.2) \quad \{B \otimes B\} = r_{12}(B \otimes B) - (B \otimes B)r_{12},$$

$$(3.3) \quad \{C \otimes C\} = r_{12}(C \otimes C) - (C \otimes C)r_{12},$$

$$(3.4) \quad \{C \otimes B\} = r_{12}(C \otimes B) - (C \otimes B)r_{12}.$$

It is a standard result that $GL_N \times GL_N$ with the above bracket is a Poisson Lie group.

We classify all quadratic Poisson brackets on $GL_N \times GL_N \times \mathcal{A}$ for which the Poisson Lie group action of $GL_N \times GL_N$ on \mathcal{A} defined by (3.1) is Poisson:

Lemma 3.1. *Provided that the brackets between B and C matrices are given by (3.2), (3.3), and (3.4), that the brackets for \mathbb{A} are given by (1.2), and that the brackets between B , C , and \mathbb{A} have the form (1.6):*

$$\begin{aligned} \{B \otimes \mathbb{A}\} &= BQ_{12}\mathbb{A} + B\mathbb{A}Q_{12}^{t_2}, \\ \{C \otimes \mathbb{A}\} &= CQ_{12}\mathbb{A} + C\mathbb{A}Q_{12}^{t_2}, \end{aligned}$$

in order for these brackets to satisfy the Jacobi identities and for the mapping $\mathbb{A} \mapsto B\mathbb{A}C^T$ to be an automorphism of the Poisson algebra (1.2), we have exactly three choices of the matrix Q_{12} itemized in Lemma 2.1.

In cases (ii) and (iii), the combination $\mathfrak{A} := B\mathbb{A}^{-T}C^T$ satisfies the same algebra (1.2) as \mathbb{A} and $B\mathbb{A}C^T$.

Proof. Straightforward calculation using the r -matrix form of writing for the corresponding brackets. \square

We again have an analogue of Lemma 2.3.

Lemma 3.2. *The transformation $\mathbb{A} \mapsto \mathbb{A}^{-1}$, $B \mapsto C^{-T}$, $C \mapsto B^{-T}$ is an anti-automorphism of the Poisson algebra for the triple (\mathbb{A}, B, C) with the interchange $Q_{12}^{(ii)} \leftrightarrow Q_{12}^{(iii)}$.*

We now interpret \mathcal{A} as a Poisson symmetric space:

Theorem 3.3. *The map*

$$(3.5) \quad \begin{aligned} \Theta : \quad GL_N \times GL_N &\rightarrow GL_N \times GL_N \\ (B, C) &\mapsto (C^{-T}, B^{-T}) \end{aligned}$$

is an involutive anti-Poisson automorphism so that $(GL_N \times GL_N, \Theta)$ is a Poisson symmetric Lie group.

Let $H \subset GL_N \times GL_N$ be the fixed point set of Θ . Then the immersion:

$$(3.6) \quad \begin{array}{ccc} i: & GL_N \times GL/H & \rightarrow \mathcal{A} \\ & (B, C) & \mapsto BC^T \end{array}$$

is a Poisson isomorphism.

Proof. The fact that Θ is an involutive automorphism is obvious. To prove that it is anti-Poisson we need to prove that $(B', C') := \Theta(B, C)$ satisfy the following Poisson brackets:

$$\begin{aligned} \{B' \otimes B'\} &= -r_{12}(B' \otimes B') + (B' \otimes B')r_{12}, \\ \{C' \otimes C'\} &= -r_{12}(C' \otimes C') + (C' \otimes C')r_{12}, \\ \{C' \otimes B'\} &= -r_{12}(C' \otimes B') + (C' \otimes B')r_{12}. \end{aligned}$$

The first two are obvious, let us prove the third:

$$\begin{aligned} \{C' \otimes B'\} &= \{B^{-T} \otimes C^{-T}\} = B^{-T} \otimes C^{-T} \{B^T \otimes C^T\} B^{-T} \otimes C^{-T} = \\ &= B^{-T} \otimes C^{-T} (-r_{21}(B \otimes C) + (B \otimes C)r_{21})^T B^{-T} \otimes C^{-T} = \\ &= -r_{21}^T B^{-T} \otimes C^{-T} + B^{-T} \otimes C^{-T} r_{21}^T = -r_{12}(C' \otimes B') + (C' \otimes B')r_{12}, \end{aligned}$$

where in the last step we have used that $r_{21} = r_{12}^T$.

Let us now consider the fixed point set $H \subset GL_N \times GL_N$ of Θ :

$$H = \{(H_1, H_2) \in GL_N \times GL_N \mid H_1 = H_2^{-T}\}.$$

Then $GL_N \times GL_N/H$ is the set of equivalence classes $(B_1, C_1) \sim (B_2, C_2)$ iff $(B_1, C_1) = (B_2 H_1, C_2 H_2)$ for some $(H_1, H_2) \in H$ and it is straightforward to prove that the immersion i is an isomorphism. To prove that this is a Poisson isomorphism we first observe that the following graph commutes:

$$\begin{array}{ccc} GL_N \times GL_N/H & \xrightarrow{i} & \mathcal{A} \\ \pi \uparrow & \nearrow_{\hat{i}} & \\ GL_N \times GL_N & & \end{array}$$

where π is the coset map which associates to each element (B, C) its right coset and $\hat{i}(B, C) = BC^T$. In [15] it is proved that there exists a unique Poisson bracket on $GL_N \times GL_N/H$ for which π is a Poisson map. So assuming we endow $GL_N \times GL_N/H$ with such a Poisson map, if we prove that \hat{i} is Poisson then i is. To prove that \hat{i} is Poisson we just need to prove that

$$\begin{aligned} \{BC^T \otimes BC^T\} &= r_{12}(BC^T \otimes BC^T) - (BC^T \otimes BC^T)r_{12} + \\ &+ BC^T r_{12}^{t_1} BC^T - BC^T r_{12}^{t_1} BC^T. \end{aligned}$$

This is again a straightforward computation which uses the r -matrix properties. \square

4. CHAINS OF B -MATRICES

We now introduce the Poisson structure on the extended space $\mathcal{A} \otimes_{k=1}^n GL_N$ of chains $(\mathbb{A}, B_1, B_2, \dots, B_n)$. We want to postulate the brackets between B_i and B_{i+1} that are compatible with the following natural (partial) multiplication operation:

$$(4.1) \quad (\mathbb{A}, B_1) \circ (B_1 \mathbb{A} B_1^T, B_2) = (\mathbb{A}, B_2 B_1),$$

and its chain analogue

$$(\mathbb{A}, B_1) \circ (B_1 \mathbb{A} B_1^T, B_2) \circ \dots \circ (B_{j-1} \dots B_1 \mathbb{A} B_1^T \dots B_{j-1}^T, B_j) = (\mathbb{A}, B_j B_{j-1} \dots B_2 B_1).$$

It is easy to see that if we impose

$$\begin{aligned} \{B_2^1 \otimes B_1^2\} &= B_2^1 Q_{12} B_1^2, \\ \{B_2^1 \otimes B_2^2\} &= r_{12} B_2^1 B_2^2 - B_2^1 B_2^2 r_{12}, \\ \{B_2^1 \otimes \mathbb{A}^2\} &= 0, \end{aligned}$$

where Q_{12} is chosen as in Theorem 2.1, then these brackets ensure that all the three mappings $(\mathbb{A}, B_1, B_2) \mapsto (\mathbb{A}, B_1)$, $(\mathbb{A}, B_1, B_2) \mapsto (\mathbb{A}, B_2 B_1)$, and $(\mathbb{A}, B_1, B_2) \mapsto (B_1 \mathbb{A} B_1^T, B_2)$ be Poisson.

In the multiple chain generalisation we can prove the following:

Lemma 4.1. *The Poisson structure compatible with the groupoid multiple product*

$$(\mathbb{A}, B_1) \circ (B_1 \mathbb{A} B_1^T, B_2) \circ \dots \circ (B_{j-1} \dots B_1 \mathbb{A} B_1^T \dots B_{j-1}^T, B_j) = (\mathbb{A}, B_j B_{j-1} \dots B_2 B_1)$$

has the following form: the brackets between \mathbb{A} are given by (1.2), the bracket between \mathbb{A} and B_1 has the form (1.6), B_k with $k > 2$ Poisson commute with \mathbb{A} , whereas the Poisson brackets between B_k are

$$(4.2) \quad \{B_k^1 \otimes B_k^2\} = r_{12} B_k^1 B_k^2 - B_k^1 B_k^2 r_{12}, \quad k = 1, \dots, j,$$

$$(4.3) \quad \begin{aligned} \{B_{k+1}^1 \otimes B_k^2\} &= B_{k+1}^1 Q_{12}^{[k]} B_k^2, \quad k = 1, \dots, j-1 \\ \{B_k^1 \otimes B_l^2\} &= 0, \quad |k-l| > 1, \end{aligned}$$

where $Q_{12}^{[k]}$ is either zero, or $-r_{12}^{t_2}$, or $r_{12}^{t_1}$. As before, we refer these three cases to as (i), (ii), and (iii).

Remark 4.2. In the relation (4.3) the $Q^{[k]}$ -matrices can be different for different k (of course, all of them must be of one of three types, (i), (ii), or (iii)). In all calculations below we always assume that $Q^{[k]} = Q^{(ii)}$ for all k . In this case, the Poisson relations are uniform, but the structure of central elements is different for odd and even j , as we shall see in Section 6. If, on the contrary, we set, say, $Q^{[2r+1]} = Q^{(iii)}$ and $Q^{[2r]} = Q^{(ii)}$, we obtain uniform expressions for central elements for the price of introducing an alternating Poisson brackets.

5. CHAINS OF (B, C) -PAIRS

We now introduce the Poisson structure on the extended space $\mathcal{A} \otimes_{k=1}^{2n} GL_N$ of chains $(\mathbb{A}, (B_1, C_1), (B_2, C_2), \dots, (B_n, C_n))$. The (partial) multiplication operation now reads

$$(5.1) \quad (\mathbb{A}, B_1, C_1) \circ (B_1 \mathbb{A} C_1^T, B_2, C_2) = (\mathbb{A}, B_2 B_1, C_2 C_1).$$

We now must postulate the brackets between B_1, C_1 and B_2, C_2 that are compatible with this multiplication (5.1). Inside every pair (B_i, C_i) , the brackets coincide with (3.2)–(3.4); it is then easy to see that if we impose

$$\begin{aligned} \{B_2 \otimes B_1\} &= B_2 Q_{12} B_1, \\ \{C_2 \otimes B_1\} &= C_2 Q_{12} B_1, \\ \{B_2 \otimes C_1\} &= B_2 Q_{12} C_1, \\ \{C_2 \otimes C_1\} &= C_2 Q_{12} C_1, \\ \{B_2 \otimes \mathbb{A}\} &= \{C_2 \otimes \mathbb{A}\} = 0, \end{aligned}$$

where Q_{12} , as above, is one of three cases (i), (ii), and (iii), then all the three mappings $(\mathbb{A}, (B_1, C_1), (B_2, C_2)) \mapsto (\mathbb{A}, B_1, C_1)$, $(\mathbb{A}, (B_1, C_1), (B_2, C_2)) \mapsto (\mathbb{A}, B_2 B_1, C_2 C_1)$, and $(\mathbb{A}, (B_1, C_1), (B_2, C_2)) \mapsto (B_1 \mathbb{A} C_1^T, B_2, C_2)$ are Poisson.

Lemma 5.1. *The Poisson structure compatible with the groupoid multiple product*

$$\begin{aligned} &(\mathbb{A}, B_1, C_1) \circ (B_1 \mathbb{A} C_1^T, B_2, C_2) \circ \dots \circ (B_{j-1} \dots B_1 \mathbb{A} C_1^T \dots C_{j-1}^T, B_j, C_j) \\ &= (\mathbb{A}, B_j B_{j-1} \dots B_2 B_1, C_j C_{j-1} \dots C_2 C_1) \end{aligned}$$

has the following form: the brackets between \mathbb{A} are given by (1.2), the bracket between \mathbb{A} and B_1, C_1 has the form as in Lemma 3.1, B_k and C_k with $k > 2$ Poisson commute with \mathbb{A} , the Poisson brackets between B_k and C_k are of the form (3.2)–(3.4) for any k , and the remaining possibly nonvanishing Poisson brackets are

$$(5.2) \quad \{B_{k+1} \otimes B_k\} = B_{k+1} Q_{12}^{[k]} B_k, \quad k = 1, \dots, j-1,$$

$$(5.3) \quad \{C_{k+1} \otimes B_k\} = C_{k+1} Q_{12}^{[k]} B_k, \quad k = 1, \dots, j-1,$$

$$(5.4) \quad \{B_{k+1} \otimes C_k\} = B_{k+1} Q_{12}^{[k]} C_k, \quad k = 1, \dots, j-1,$$

$$(5.5) \quad \{C_{k+1} \otimes C_k\} = C_{k+1} Q_{12}^{[k]} C_k, \quad k = 1, \dots, j-1,$$

where $Q_{12}^{[k]}$ is either zero, or $-r_{12}^{t_2}$, or $r_{12}^{t_1}$. As before, we refer these three cases to as (i), (ii), and (iii).

In the above relations, like in the case of chains of B -matrices, the matrices $Q^{[k]}$ can be different for different k . In all calculation below we always assume that $Q^{[k]} = Q^{(ii)}$ for all k .

6. CENTRAL ELEMENTS

6.1. Central elements of the \mathbb{A} - and B -matrix algebras. The central elements of the Poisson algebra (1.1) of $a_{i,j}$ are of two types: we have the polynomial central elements and rational central elements; together they form a set of exactly N algebraically independent central elements.

The *polynomial central elements* r_k are given by the coefficients of λ^{-k} , $k = 0, 1, \dots, [N/2] + 1$, of the polynomial

$$\det(\mathbb{A} + \lambda^{-1} \mathbb{A}^T).$$

The *rational central elements* are defined by the *bottom-left minors* of the general matrix \mathbb{A} (provided these minors do not vanish): we take

$$M_d^-(A) := \det \begin{pmatrix} a_{N-d+1,1} & \dots & a_{N+d-1,d} \\ \vdots & \dots & \vdots \\ a_{N,1} & \dots & a_{N,d} \end{pmatrix}.$$

In [3] (see also [2] where these elements were found to be central for symmetric \mathbb{A}), we have proved that for every $d = 1, \dots, [\frac{N}{2}]$ the quantities

$$b_d := M_d^-(A)/M_{N-d}^-(A)$$

are central elements of the Poisson algebra (1.1).

The central elements of the Poisson algebra (2.3) (see [9]) are generated by the complementary minors: let

$$(6.1) \quad M_d^-(B) := \det \begin{pmatrix} b_{N-d+1,1} & \dots & b_{N+d-1,d} \\ \vdots & \dots & \vdots \\ b_{N,1} & \dots & b_{N,d} \end{pmatrix}$$

and

$$(6.2) \quad M_d^+(B) := \det \begin{pmatrix} b_{1,N-d} & \dots & b_{1,N} \\ \vdots & \dots & \vdots \\ b_{d,N-d} & \dots & b_{d,N} \end{pmatrix}$$

be the minors located at the respective bottom-left and upper-right corners of the matrix B . We then have exactly N algebraically independent central elements

$$c_d = M_d^+(B)/M_{N-d}^-(B), \quad d = 1, \dots, N.$$

Note that these are exactly the minors that appeared in [1] in the structure of B -matrices for the groupoid of upper-triangular matrices.

6.2. Casimir functions of the Lie–Poisson brackets for the B and C matrices.

Lemma 6.1. *The Poisson–Lie brackets (3.2)–(3.4) of the (B, C) -system possess $2N$ Casimir functions:*

- (i) $N + 1$ Casimir functions generated by the minors of the matrices B and C (in the notation of (6.1)) and (6.2)

$$(6.3) \quad M_d^-(B)/M_{N-d}^+(C), \quad d = 0, 1, \dots, N-1, N;$$

- (ii) $N+1$ Casimir functions q_s generated by the coefficients of λ^s of the expansion of

$$(6.4) \quad \det(B + \lambda C) = \sum_{s=0}^N \lambda^s q_s.$$

Note that $\det C$ and $\det B$ enter the both sets, so the total number of algebraically independent Casimir functions is exactly $2N$.

Proof. That the above elements are central is a relatively easy calculation. A more lengthy is the proof that the general degeneracy of the Poisson brackets (3.2)–(3.4) is $2N$. To prove it, we first consider the linearized algebraic version of these brackets for $B = \mathbb{E} + \epsilon b$ and $C = \mathbb{E} + \epsilon c$:

$$(6.5) \quad \{b \otimes b\} = r_{12}(\mathbb{E} \otimes b + b \otimes \mathbb{E}) - (\mathbb{E} \otimes b + b \otimes \mathbb{E})r_{12},$$

$$(6.6) \quad \{c \otimes c\} = r_{12}(\mathbb{E} \otimes c + c \otimes \mathbb{E}) - (\mathbb{E} \otimes c + c \otimes \mathbb{E})r_{12},$$

$$(6.7) \quad \{c \otimes b\} = r_{12}(\mathbb{E} \otimes b + c \otimes \mathbb{E}) - (\mathbb{E} \otimes b + c \otimes \mathbb{E})r_{12},$$

We are going to solve the linear system of equations w.r.t. $N \times N$ -matrices x and y

$$(6.8) \quad \{b \otimes \text{tr}_2(bx + cy)\} = \{c \otimes \text{tr}_2(bx + cy)\} = 0.$$

Using the explicit form of the r -matrix, these two systems of equations can be written in the form (here $P_{+,1/2}$ and $P_{-,1/2}$ are the standard projection operators)

$$(6.9) \quad \begin{aligned} &P_{+,1/2}(x)b + P_{+,1/2}(bx) - bP_{+,1/2}(x) - P_{+,1/2}(xb) \\ &- P_{-,1/2}(y)b - P_{-,1/2}(cy) + bP_{-,1/2}(y) + P_{-,1/2}(yc) = 0, \end{aligned}$$

$$(6.10) \quad \begin{aligned} &P_{+,1/2}(bx) + P_{+,1/2}(x)c - cP_{+,1/2}(x) - P_{+,1/2}(xc) \\ &+ P_{+,1/2}(cy) + P_{+,1/2}(y)c - cP_{+,1/2}(y) - P_{+,1/2}(yc) = 0. \end{aligned}$$

Subtracting the second equation from the first one, we obtain a simple restriction that

$$(6.11) \quad [P_{+,1/2}(x) - P_{-,1/2}(y), b - c] = 0.$$

We now choose the matrices b and c in the special form containing only diagonal and anti-diagonal parts, $b_{i,j} = b_i \delta_{i,j} + \delta_{i,N+1-i}$ and $c_{i,j} = c_i \delta_{i,j} + \delta_{i,N+1-i}$ with all b_i and c_i distinct. It is then easy to see that among all non-diagonal entries of x and y only the entries on the lower half-anti-diagonal of x and on the upper half-anti-diagonal of y can be nonzero and substituting this ansatz into (6.9) we obtain exactly N equations

$$\begin{aligned} x_{d,N+1-d} + y_{N+1-d,d} &= 0, \quad d = 1, \dots, [N/2], \\ 1/2(x_{d,d} + y_{d,d}) &= 1/2(x_{N+1-d,N+1-d} + y_{N+1-d,N+1-d}) \end{aligned}$$

on $3N$ variables. This clearly indicates that we have exactly $2N$ -dimensional space of solutions corresponding to $2N$ Casimir functions. The lemma is proved. \square

6.3. The Casimir functions of the type-(ii) (\mathbb{A}, B) -system. We begin constructing Casimir functions for algebra (ii) by noting that the brackets (1.2) and (1.6) coincide for \mathbb{A} and \mathbb{A}^T , that is,

$$(6.12) \quad \{B \otimes (\mathbb{A} + \lambda \mathbb{A}^T)\} = B Q_{12} (\mathbb{A} + \lambda \mathbb{A}^T) + B (\mathbb{A} + \lambda \mathbb{A}^T) Q_{12}^T$$

for any λ and for any choice of the r -matrix Q . It then follows, in the case of bracket (ii), that

$$(6.13) \quad \{b_{ij}, \det(\mathbb{A} + \lambda \mathbb{A}^T)\} = -2b_{ij} \det(\mathbb{A} + \lambda \mathbb{A}^T),$$

and, recalling that every $\det(\mathbb{A} + \lambda \mathbb{A}^T)$ is a Casimir function of the \mathbb{A} -algebra, we obtain that the elements

$$(6.14) \quad \frac{\det(\mathbb{A} + \lambda \mathbb{A}^T)}{\det \mathbb{A}}$$

are Casimir functions of the total algebra.

To construct the other set of Casimir functions we recall the combination \mathfrak{A} introduced in (2.5).

Remark 6.2. The matrix \mathfrak{A} has the following Poisson relations with \mathbb{A} and B :

$$(6.15) \quad \{B \otimes \mathfrak{A}\} = r_{12} B \mathfrak{A} + \mathfrak{A} r_{12}^T B,$$

$$(6.16) \quad \{\mathfrak{A} \otimes \mathbb{A}\} = -2B(P_{12}^{t_1} + \mathbb{A}^{-T} \mathbb{A} P_{12}) B^T.$$

We see that the second bracket destroys the structure of \mathfrak{A} . However, if we introduce the combination

$$(6.17) \quad S := \mathbb{A}^T B^{-1}$$

we observe that this quantity does have consistent brackets with \mathbb{A} , B , and itself:

$$(6.18) \quad \{S \otimes B\} = S r_{12}^{t_1 t_2} B + B r_{12}^T S,$$

$$(6.19) \quad \{S \otimes \mathbb{A}\} = r_{12} S \mathbb{A} + \mathbb{A} r_{12}^T S,$$

$$(6.20) \quad \{S \otimes S\} = r_{12} S S - S r_{12}^{t_1 t_2}.$$

These formulas are instrumental when finding Casimir functions for the brackets of type (ii).

We now use the algebra (6.18–6.20) of S -variables (6.17) to study the Casimirs. We first write formulas (6.18), (6.19), the Poisson relation between $b_{i,j}$ and $a_{k,l}$, and the brackets between b 's and s 's in components:

$$(6.21) \quad \{s_{i,j}, b_{k,l}\} = \sum_{\rho=1}^m s_{i,\rho} b_{\rho,l} \theta(j - \rho) \delta_{j,k} + s_{l,j} b_{k,i} \theta(i - l),$$

$$(6.22) \quad \{s_{i,j}, a_{k,l}\} = s_{k,j} a_{i,l} \theta(i - k) + s_{l,j} a_{k,i} \theta(i - l),$$

$$(6.23) \quad \{b_{i,j}, a_{k,l}\} = -b_{i,k} a_{j,l} \theta(k - j) - b_{i,l} a_{k,j} \theta(l - j),$$

$$(6.24) \quad \{b_{i,j}, b_{k,l}\} = b_{i,l} b_{k,j} (\theta(i - k) - \theta(l - j)),$$

$$(6.25) \quad \{s_{i,j}, s_{k,l}\} = s_{i,l} s_{k,j} (\theta(i - k) - \theta(j - l)),$$

We now let M_B^p denote the $(p \times p)$ -minor of the matrix B located at the upper-right corner and let M_S^p denote the $(p \times p)$ -principal minor of the matrix S (located at the upper-left corner).

Using the same technique as in Sec. 6.1, we can demonstrate that the above relations (6.21)–(6.24) imply that all the brackets of $a_{i,j}$ and $b_{i,j}$ with M_S^p and M_B^p are :

$$\begin{aligned} \{M_S^p, a_{i,j}\} &= [D^p]_{i,j} \cdot M_S^p a_{i,j}, \\ \{M_S^p, b_{i,j}\} &= [D^p]_{i,j} \cdot M_S^p b_{i,j}, \\ \{M_B^p, a_{i,j}\} &= [F^p]_{i,j} \cdot M_B^p a_{i,j}, \\ \{M_B^p, b_{i,j}\} &= [G_0^p]_{i,j} \cdot M_B^p b_{i,j}, \end{aligned}$$

where D^p, F^p, G_0^p are integer-valued matrices composed of blocks in which all entries are the same. We represent them graphically as follows:

$$(6.26) \quad D^p = \begin{array}{c} p \\ \begin{array}{|c|c|c|} \hline 2 & 1 & 1 \\ \hline 1 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline \end{array} \end{array}, \quad F^p = \begin{array}{c} p \\ \begin{array}{|c|c|c|} \hline 0 & 0 & -1 \\ \hline 0 & 0 & -1 \\ \hline -1 & -1 & -2 \\ \hline \end{array} \end{array}, \quad G_0^p = \begin{array}{c} p \\ \begin{array}{|c|c|c|} \hline 1 & 1 & 0 \\ \hline 0 & 0 & -1 \\ \hline 0 & 0 & -1 \\ \hline \end{array} \end{array},$$

where we assume that the unit squares are $(p \times p)$ -blocks (and that $N - p$ is visually twice bigger than p) and that each block is composed by a matrix with all entries equal to the given integer (for example the $p \times p$ block 2 is a $p \times p$ matrix with all entries equal to 2).

From this graphical representation we immediately obtain that the combination M_S^p/M_B^{N-p} has constant brackets with all $a_{i,j}$ for any p :

$$(6.27) \quad \left\{ \frac{M_S^p}{M_B^{N-p}}, a_{i,j} \right\} = [D^p - F^{N-p}]_{i,j} \frac{M_S^p}{M_B^{N-p}} a_{i,j} = 2 \frac{M_S^p}{M_B^{N-p}} a_{i,j}, \quad \forall p,$$

This ratio of minors does not still have constant brackets with $b_{i,j}$, but if we take the *product* of two such ratios, then we finally obtain the constant relation for any i and j :

$$\left\{ \frac{M_S^p M_S^{N-p}}{M_B^{N-p} M_B^p}, b_{i,j} \right\} = [D^p + D^{N-p} - G_0^p - G_0^{N-p}]_{i,j} \frac{M_S^p M_S^{N-p}}{M_B^{N-p} M_B^p} b_{i,j} = 2 \frac{M_S^p M_S^{N-p}}{M_B^{N-p} M_B^p} b_{i,j}.$$

Now in order to obtain commuting quantities it suffices to multiply this expression by the appropriate powers of the determinants $\det \mathbb{A}$ and $\det B$. We have therefore proved the following theorem.

Theorem 6.3. *We have the following $2 \lfloor \frac{N}{2} \rfloor$ (algebraically independent) Casimir functions of the Poisson brackets (ii):*

- $\lfloor \frac{N}{2} \rfloor$ coefficients Y_p of λ -expansion

$$\frac{\det(\lambda \mathbb{A} + \lambda^{-1} \mathbb{A}^T)}{\det \mathbb{A}} = (\lambda^N + \lambda^{-N}) + \sum_{p=1}^{\lfloor N/2 \rfloor} (\lambda^{N-2p} + \lambda^{2p-N}) Y_p;$$

- $\lfloor \frac{N}{2} \rfloor$ Casimir functions

$$X_p := \frac{M_S^p M_S^{N-p}}{M_B^{N-p} M_B^p} \cdot \frac{[\det B]^2}{\det \mathbb{A}}, \quad p = 1, \dots, \lfloor N/2 \rfloor,$$

where M_S^q are the $(q \times q)$ -principal (situated at the upper-left corner) minors of the matrix $S = \mathbb{A}^T B^{-1}$ and M_B^q are $(q \times q)$ -minors of the matrix B situated at the upper-right corner.

Remark 6.4. Note that both Y_p and X_p are functions of $a_{i,j}$ and $b_{i,j}$ of degree zero: scaling independently all $b_{i,j}$ and all $a_{i,j}$ does not change the values of the Casimir functions.

6.4. The Casimir functions of the type-(ii) (\mathbb{A}, B, C) -triple. It is again useful to consider the same combination $S = \mathbb{A}^T B^{-1}$ as in the previous section. We now need its commutation relations with C :

$$(6.28) \quad \{S \otimes C\} = {}^1 S r_{12}^{t_1 t_2} {}^2 C + {}^2 C r_{12}^{t_2} {}^1 S,$$

We see that they have exactly the same form as (6.18), so it is not surprising that the principal minors of the S -matrix have commutation relations with entries of the C matrix as well. On the other hand, it is now minors of the matrix C , not B , located at the upper-right corner that have commutation relations. Applying the same technique as in the proof of Theorem 6.3, we come to the following statement.

Theorem 6.5. *We have the following $2 \lfloor \frac{N}{2} \rfloor + N$ (algebraically independent) Casimir functions of the Poisson brackets (ii) in the case of (\mathbb{A}, B, C) -triple:*

- $\lfloor \frac{N}{2} \rfloor$ coefficients Y_p of λ -expansion

$$\frac{\det(\lambda \mathbb{A} + \lambda^{-1} \mathbb{A}^T)}{\det \mathbb{A}} = (\lambda^N + \lambda^{-N}) + \sum_{p=1}^{\lfloor N/2 \rfloor} (\lambda^{N-2p} + \lambda^{2p-N}) Y_p;$$

- $\lfloor \frac{N}{2} \rfloor$ Casimir functions

$$X_p := \frac{M_S^p M_S^{N-p}}{M_C^{N-p} M_C^p} \cdot \frac{\det B \det C}{\det \mathbb{A}}, \quad p = 1, \dots, \lfloor N/2 \rfloor,$$

where M_S^q are the $(q \times q)$ -principal (situated at the upper-left corner) minors of the matrix $S = \mathbb{A}^T B^{-1}$ and M_C^q are $(q \times q)$ -minors of the matrix C situated at the upper-right corner;

- N coefficients Z_p of the λ -expansion

$$\det(B + \lambda C) / \det B = 1 + \sum_{p=1}^N \lambda^p Z_p.$$

6.5. Casimir functions for chains of B -matrices. As above, we mainly consider the case (ii). In this case, for a chain of matrices B_1, \dots, B_j we again have the special matrix S whose form will be different for odd and even j :

$$(6.29) \quad S = \begin{cases} \mathbb{A}^T B_1^{-1} B_2^T B_3^{-1} \dots B_{j-1}^T B_j^{-1} & \text{for odd } j, \\ \mathbb{A}^T B_1^{-1} B_2^T B_3^{-1} \dots B_{j-1}^{-1} B_j^T & \text{for even } j. \end{cases}$$

We now consider the case $j > 1$. The matrix S has the same Poisson relations with \mathbb{A} and B_1 irrespectively whether j is odd or even:

$$(6.30) \quad \{S \otimes \mathbb{A}\} = r_{12} {}^1 S \mathbb{A} + {}^2 \mathbb{A} r_{12}^{t_2} {}^1 S$$

$$(6.31) \quad \{S \otimes B_1\} = {}^2 B_1 r_{12}^{t_2} {}^1 S$$

Next, it is easy to see that S commutes with all B_k , $k = 2, \dots, j-1$. This follows from the following observation:

$$\begin{aligned}
\{B_k^1 \otimes B_{k-1}^T B_k^{-2} B_{k+1}^T\} &= \{B_k^1 \otimes B_{k-1}^T\} B_k^{-2} B_{k+1}^T \\
&+ B_{k-1}^T \{B_k^1 \otimes B_k^{-2}\} B_{k+1}^T + B_{k-1}^T B_k^{-2} \{B_k^1 \otimes B_{k+1}^T\} \\
&= -B_k^1 B_{k-1}^T r_{12} B_k^{-2} B_{k+1}^T + B_{k-1}^T \left(-B_k^{-2} r_{12} B_k^1 + B_k^1 r_{12} B_k^{-2} \right) B_{k+1}^T \\
&+ B_{k-1}^T B_k^{-2} r_{12} B_k^1 B_{k+1}^T \\
&= 0.
\end{aligned}$$

The only difference occurs in the last commutation relations:

- for j odd we have that

$$\{S^1 \otimes B_j^{-2}\} = -S^1 B_j^{-2} r_{12}^{t_1 t_2}$$

and

$$\{S^1 \otimes S^2\} = r_{12} S^1 S^2 - S^1 S^2 r_{12}^{t_1 t_2}$$

- for j even we have that

$$\{S^1 \otimes B_j^T\} = -S^1 B_j^T r_{12}$$

and

$$\{S^1 \otimes S^2\} = r_{12} S^1 S^2 - S^1 S^2 r_{12}$$

As in Sec. 6.3, we can construct commuting elements from the corresponding minors of S and B_k . But now we again observe the difference between the cases of odd and even j .

Notation 6.6. We let M_S^p denote the *principal* (located at the upper-left corner) $(p \times p)$ -minor of the matrix S (6.29) for *odd* j and let the same symbol M_S^p denote the *upper-right* $(p \times p)$ -minor of the matrix S (6.29) for *even* j . For all the matrices B_k we let $M_{B_k}^p$ denote their *upper-right* $(p \times p)$ -minors.

As above, all these minors have commutation relations with all $a_{s,q}$ and $[b_k]_{s,q}$. Below we present all nonzero commutation relations:

$$\begin{aligned}
\{M_S^p, a_{s,q}\} &= [D^p]_{s,q} M_S^p a_{s,q}, \\
\{M_S^p, [b_1]_{s,q}\} &= [G_-^p]_{s,q} M_S^p [b_1]_{s,q}, \\
\{M_S^p, [b_j]_{s,q}\} &= \begin{cases} [G_+^p]_{s,q} M_S^p [b_j]_{s,q} & \text{for even } j, \\ [E^p]_{s,q} M_S^p [b_j]_{s,q} & \text{for odd } j, \end{cases} \\
\{M_{B_1}^p, a_{s,q}\} &= [F^p]_{s,q} M_{B_1}^p a_{s,q}, \\
\{M_{B_k}^p, [b_k]_{s,q}\} &= [G_0^p]_{s,q} M_{B_k}^p [b_k]_{s,q}, \quad k = 1, \dots, j, \\
\{M_{B_k}^p, [b_{k+1}]_{s,q}\} &= [G_-^p]_{s,q} M_{B_k}^p [b_{k+1}]_{s,q}, \quad k = 1, \dots, j-1, \\
\{M_{B_k}^p, [b_{k-1}]_{s,q}\} &= [G_+^p]_{s,q} M_{B_k}^p [b_{k-1}]_{s,q}, \quad k = 2, \dots, j.
\end{aligned}$$

Here the matrices D^p , F^p , and G_0^p are defined in (6.26), and the remaining three matrices have the form

$$(6.32) \quad G_-^p = \begin{array}{c} p \\ \begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline \end{array} \end{array}, \quad G_+^p = \begin{array}{c} p \\ \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline -1 & -1 & -1 \\ \hline \end{array} \end{array}, \quad E^p = \begin{array}{c} p \\ \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \end{array}.$$

Introducing the combination

$$(6.33) \quad K^p = \begin{cases} \frac{M_S^p}{M_{B_1}^{N-p}} \frac{M_{B_2}^p}{M_{B_3}^{N-p}} \cdots \frac{M_{B_{j-2}}^p}{M_{B_{j-1}}^{N-p}} M_{B_j}^p & \text{for even } j, \\ \frac{M_S^p}{M_{B_1}^{N-p}} \frac{M_{B_2}^p}{M_{B_3}^{N-p}} \cdots \frac{M_{B_{j-1}}^p}{M_{B_j}^{N-p}} & \text{for odd } j, \end{cases}$$

we see that it has zero brackets with all B_k , $k = 1, \dots, j-1$, it has the constant bracket

$$\{K^p, a_{s,p}\} = 2K^p a_{s,p}$$

with entries of the matrix \mathbb{A} , whereas its brackets with B_j are again different depending on case of which j , odd or even, we are dealing with:

$$(6.34) \quad \{K^p, [b_j]_{s,p}\} = \begin{cases} [G_+^p - G_-^{N-p} + G_0^p]_{s,p} K^p [b_j]_{s,p} & \text{for even } j, \\ [E^p + G_-^{N-p} - G_0^{N-p}]_{s,p} K^p [b_j]_{s,p} & \text{for odd } j. \end{cases}$$

In the case of even j , the matrix combination $G_+^p - G_-^{N-p} + G_0^p$ has the form

$$G_+^p - G_-^{N-p} + G_0^p = \begin{array}{c} p \\ \begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline -1 & -1 & -1 \\ \hline -2 & -2 & -2 \\ \hline \end{array} \end{array}$$

for $p < \lfloor \frac{m}{2} \rfloor$, and it has exactly the same form if we replace p by $N-p$, so

$$[G_+^p - G_-^{N-p} + G_0^p] - [G_+^{N-p} - G_-^p + G_0^{N-p}] = 0,$$

and we obtain that the ratio K^p/K^{N-p} is truly central for any $p = 0, \dots, N$. Note that the total number of such algebraically independent combinations is $\lfloor (N+1)/2 \rfloor$, so, together with $\lfloor N/2 \rfloor$ Casimir functions generated by $\det(\mathbb{A} + \lambda \mathbb{A}^T)/\det \mathbb{A}$, we obtain exactly N algebraically independent Casimir functions.

In the case of odd j , the combination $E^p + G_-^{N-p} - G_0^{N-p}$ has the form

$$\begin{array}{c} p \\ \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \end{array}$$

(the upper and lower strips have widths equal p) in the case where $p < \lfloor N/2 \rfloor$ and

$$\begin{array}{c} N-p \\ \begin{array}{|c|c|c|} \hline 1 & 1 & 1 \\ \hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \end{array} \quad \text{(the upper and lower strips have widths equal } N-p \text{) in}$$

the case where $p > \lfloor N/2 \rfloor$. It is now the *product* $K^p K^{N-p}$, not the ratio, that has constant brackets with all a 's and b_k 's:

$$\{K^p K^{N-p}, a_{s,p}\} = 4K^p K^{N-p} a_{s,p}, \quad \{K^p K^{N-p}, [b_j]_{s,p}\} = 2K^p K^{N-p} [b_j]_{s,p},$$

and all other brackets vanish. We again (as in Sec. 6.3) must multiply by the proper combination of determinants of \mathbb{A} and B_k (all these determinants have constant brackets with all a 's and b_k 's) in order to obtain the true Casimir functions. Their number is $\lfloor N/2 \rfloor$. We collect together the obtained results in the following theorem.

Theorem 6.7. *We have the following (algebraically independent) Casimir functions of the Poisson brackets (ii) in the case of the chain of matrices $\mathbb{A}, B_1, B_2, \dots, B_j$:*

- **for all j we have $\lfloor \frac{N}{2} \rfloor$ coefficients Y_p of λ -expansion**

$$\frac{\det(\lambda \mathbb{A} + \lambda^{-1} \mathbb{A}^T)}{\det \mathbb{A}} = (\lambda^N + \lambda^{-N}) + \sum_{p=1}^{\lfloor N/2 \rfloor} (\lambda^{N-2p} + \lambda^{2p-N}) Y_p;$$

- **in the case of even j we have $\lfloor \frac{N+1}{2} \rfloor$ Casimir functions**

$$X_p = \frac{M_S^p \prod_{k=1}^j M_{B_k}^p}{M_S^{N-p} \prod_{k=1}^j M_{B_k}^{N-p}}$$

where $M_{B_k}^q$ are $(q \times q)$ -upper-right minors of the matrices B_k and M_S^q are the $(q \times q)$ -upper-right minors of the matrix $S = \mathbb{A}^T B_1^{-1} B_2^T \cdots B_{j-1}^{-1} B_j^T$;

- **in the case of odd $j = 2r + 1$ we have $\lfloor \frac{N}{2} \rfloor$ Casimir functions**

$$X_p := \frac{M_S^p M_S^{N-p} \prod_{k=1}^r (M_{B_{2k}}^p M_{B_{2k}}^{N-p})}{\prod_{k=1}^{r+1} (M_{B_{2k-1}}^p M_{B_{2k-1}}^{N-p})} \cdot \frac{\prod_{k=1}^{r+1} [\det B_{2k-1}]^2}{\det \mathbb{A} \prod_{k=1}^r [\det B_{2k}]^2},$$

where $M_{B_k}^q$ are $(q \times q)$ -upper-right minors of the matrices B_k and M_S^q are now the $(q \times q)$ -principal (situated at the upper-left corner) minors of the matrix $S = \mathbb{A}^T B_1^{-1} B_2^T \cdots B_{j-1}^T B_j^{-1}$.

We therefore have $\lfloor \frac{N}{2} \rfloor + \lfloor \frac{N+1}{2} \rfloor = N$ Casimir functions for even j and $2\lfloor \frac{N}{2} \rfloor$ Casimir functions for odd j .

Remark 6.8. Note that the statement of Theorem 6.7 remains valid both for $j = 0$ and $j = 1$. In the first case, we obtain all the Casimir functions of the \mathbb{A} -matrix algebra (1.2) and in the second case we reproduce the statement of Theorem 6.3.

6.6. Casimir functions for chains of (B, C) -pairs. As above, we consider the case (ii). In this case, for a chain of (B, C) -pairs $(B_1, C_1), \dots, (B_j, C_j)$ we again have the special matrix S whose form is different for odd and even j ; this matrix has exactly the form (6.29), that is, it is *independent* on matrices C_k .

The subsequent reasonings are similar to those presented in preceding subsections; we formulate only the final statement.

Notation 6.9. We let M_S^p denote the *principal* (located at the upper-left corner) $(p \times p)$ -minor of the matrix S (6.29) for *odd* j and let the same symbol M_S^p denote the *upper-right* $(p \times p)$ -minor of the matrix S (6.29) for *even* j . For all the matrices B_k we let $M_{B_k}^p$ denote their *upper-right* $(p \times p)$ -minors.

Theorem 6.10. *The following are (algebraically independent) Casimir functions of the Poisson brackets (ii) in the case of the chain of matrices $\mathbb{A}, (B_1, C_1), \dots, (B_j, C_j)$:*

- **for all j we have $\lfloor \frac{N}{2} \rfloor$ coefficients Y_p of λ -expansion**

$$\frac{\det(\lambda \mathbb{A} + \lambda^{-1} \mathbb{A}^T)}{\det \mathbb{A}} = (\lambda^N + \lambda^{-N}) + \sum_{p=1}^{\lfloor N/2 \rfloor} (\lambda^{N-2p} + \lambda^{2p-N}) Y_p;$$

- for all j we have $N \cdot j$ coefficients $Z_p^{(k)}$ of λ -expansion

$$\frac{\det(B_k + \lambda C_k)}{\det B_k} = 1 + \sum_{p=1}^N \lambda^p Z_p^{(k)}, \quad k = 1, \dots, j;$$

- in the case of even j we have $\lfloor \frac{N+1}{2} \rfloor$ Casimir functions

$$X_p = \frac{M_S^p \prod_{k=1}^j M_{C_k}^p}{M_S^{N-p} \prod_{k=1}^j M_{C_k}^{N-p}}$$

where $M_{C_k}^q$ are $(q \times q)$ -upper-right minors of the matrices C_k and M_S^q are the $(q \times q)$ -upper-right minors of the matrix $S = \mathbb{A}^T B_1^{-1} B_2^T \dots B_{j-1}^{-1} B_j^T$;

- in the case of odd $j = 2r + 1$ we have $\lfloor \frac{N}{2} \rfloor$ Casimir functions

$$X_p := \frac{M_S^p M_S^{N-p} \prod_{k=1}^r (M_{C_{2k}}^p M_{C_{2k}}^{N-p})}{\prod_{k=1}^{r+1} (M_{C_{2k-1}}^p M_{C_{2k-1}}^{N-p})} \cdot \frac{\prod_{k=1}^{r+1} [\det B_{2k-1} \det C_{2k-1}]}{\det \mathbb{A} \prod_{k=1}^r [\det B_{2k} \det C_{2k}]},$$

where M_S^q are now the $(q \times q)$ -principal (situated at the upper-left corner) minors of the matrix $S = \mathbb{A}^T B_1^{-1} B_2^T \dots B_{j-1}^T B_j^{-1}$ and $M_{C_k}^q$ are $(q \times q)$ -upper-right minors of the matrices C_k .

We therefore have $\lfloor \frac{N}{2} \rfloor + \lfloor \frac{N+1}{2} \rfloor + Nj = N(j+1)$ Casimir functions for even j and $2\lfloor \frac{N}{2} \rfloor + Nj$ Casimir functions for odd j ; because the total number of matrix entries is $N^2(2j+1)$ it is easy to see that the highest dimension of Poisson leaves is always even.

As in the case of the B -matrix chains, the statement of Theorem 6.10 remains valid both for $j = 0$ and $j = 1$.

Remark 6.11. In the case (iii), the statement of Proposition 2.2 remains valid as stated, the analogue of the matrix S is $\mathfrak{S} := \mathbb{A}B^{-1}$, and this matrix has the following Poisson relations:

$$(6.35) \quad \{\mathfrak{S} \otimes B\} = -\mathfrak{S} r_{12} \overset{2}{B} - \overset{2}{B} r_{12}^t \overset{1}{\mathfrak{S}},$$

$$(6.36) \quad \{\mathfrak{S} \otimes \mathbb{A}\} = -r_{12}^{t_1 t_2} \overset{1}{\mathfrak{S}} \overset{2}{\mathbb{A}} - \overset{2}{\mathbb{A}} r_{12}^{t_1} \overset{1}{\mathfrak{S}},$$

$$(6.37) \quad \{\mathfrak{S} \otimes \mathfrak{S}\} = r_{12} \overset{1}{\mathfrak{S}} \overset{2}{\mathfrak{S}} - \overset{1}{\mathfrak{S}} \overset{2}{\mathfrak{S}} r_{12}^{t_1 t_2}.$$

The statement of Theorem 6.7 also remains valid provided we make the following changes: we replace the matrix S by the matrix \mathfrak{S} , which has the form $\mathfrak{S} = \mathbb{A}B_1^{-1}B_2^T \dots B_{j-1}^{-1}B_j^T$ for even j and $\mathfrak{S} = \mathbb{A}B_1^{-1}B_2^T \dots B_{j-1}^T B_j^{-1}$ for odd j , the minors $M_{\mathfrak{S}}^q$ are now *bottom-right* principal minors for odd j (including $j = 1$) and are *lower-left* minors for even j (including $j = 0$), whereas all the minors $M_{B_k}^q$ of all the matrices B_k are *lower-left* minors.

For the *triple* \mathbb{A}, B, C , the central elements X_p in Theorem 6.5 have now the form

$$X_p := \frac{M_{\mathfrak{S}}^p M_{\mathfrak{S}}^{N-p}}{M_B^{N-p} M_B^p} \cdot \frac{\det B \det C}{\det \mathbb{A}}, \quad p = 1, \dots, \lfloor N/2 \rfloor,$$

where $M_{\mathfrak{S}}^q$ are the *bottom-right* minors of the new matrix $\mathfrak{S} := \mathbb{A}C^{-1}$, and M_B^q are the *lower-left* minors of the matrix B .

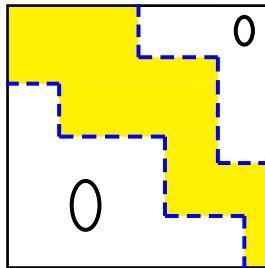


Figure 1 shows two square domains, B and C , illustrating the definition of the function f . Domain B is a square with a white region in the bottom-left corner, and domain C is a square with a white region in the top-right corner. Both white regions are bounded by a blue dashed line. The label '0' is placed inside the white region of each domain.

7. POISSON AND DIRAC REDUCTIONS

Lemma 7.1. *Any reduction depicted in Fig. 1 where all elements below the lower broken line that goes as in the figure and all elements above the second broken line are set to be zeros is a Poisson reduction of algebra (2.3).*

Lemma 7.2. *Any reduction depicted in Fig. 2 where all elements of the matrix B that are below the broken line and all elements of the matrix C that are above the broken line are set to be zeros is a Poisson reduction of algebra (3.2)-(3.4).*

Lemma 7.3. *The reduction of the (\mathbb{A}, B, C) – triple to the (\mathbb{A}, B) -double realized by imposing the constraint $B = C$ is Poisson for any choice of the matrix Q_{12} .*

Remark 7.4. Note that the above Poisson reductions of B and C matrices are compatible with Poisson reductions of \mathbb{A} -algebra (1.1): in case of Lemma 7.2 if the matrices B and C^T have a b.u.t. form, then the matrix $\mathbb{A} = BC^T$ will have the same b.u.t. form, which is known to be a Poisson reduction of the algebra (1.1); in case of Lemma 7.3, choosing $B = C$ corresponds to restricting \mathbb{A} to a symmetric form, which is again a Poisson reduction of algebra (1.1).

7.3. Restriction to the groupoid of b.u.t. matrices. We now implement the standard procedure of Dirac reduction [6] (which is of extensive use in quantum field theory, see, e.g., [22]) for constructing brackets on manifolds defined by constraints of the second kind.

Let us recall this procedure here: given a Poisson manifold of dimension M , denote the coordinates by X_i , $i = 1, \dots, M$. Choose a set of $2N$ constraints $C_k(X)$ of the second kind (i.e. such that their mutual Poisson brackets restricted to the constraint manifold are not all zero), if the matrix $D_{k,l}(X) := \{C_k, C_l\}$ is nondegenerate on the constraint surface $C_k(X) = 0$, $k = 1, \dots, 2N$, then the following new bracket called the *Dirac bracket* on the constrained manifold:

$$(7.1) \quad \{X_i, X_j\}_D \Big|_{C_k=0} := \{X_i, X_j\} - \{X_i, C_k\} D_{k,l}^{-1}(X) \{X_j, C_l\},$$

where, as usual, we imply summations over repeated indices and $D_{k,l}^{-1}$ denotes the k, l entry of D^{-1} , defines a Poisson structure on the constraint manifold.

One, truly remarkable, property of the Dirac brackets is that, provided the matrix $D_{k,l}$ be nondegenerate, the Dirac bracket (7.1) does not depend on an actual parameterization of the constraint submanifold $\{C_k = 0, k = 1, \dots, 2N\}$: taking instead of C_k arbitrary functions $f_k(\{C_i\})$ such that $f_k(\{0\}) = 0$ and $\det [\partial f_k / \partial C_k] \Big|_{C=0} \neq 0$ we obtain the *same* Dirac bracket on the constraint manifold.

We now apply the Dirac bracket procedure to the b.u.t. case. In this case, the constraints are that both \mathbb{A} and BAB^T are of the b.u.t. form:

$$(7.2) \quad \mathbb{A}_{I,J} = 0 \text{ for } I > J, \quad \det \mathbb{A}_{I,I} = 1$$

$$(7.3) \quad (BAB^T)_{I,J} = 0 \text{ for } I > J, \quad \det BAB^T_{I,I} = 1.$$

Observe, first, that since the brackets both between the elements of \mathbb{A} and between the elements of BAB^T are given by the algebra (1.1) and the above b.u.t. form is a Poisson reduction of this algebra, the brackets between \mathbb{A} -constraints vanish on the constraint surface (7.2) and the same is true for the brackets between (BAB^T) -constraints on the constraint surface (7.3). So, the only nonzero brackets on the constraint surface can be an inter-brackets between \mathbb{A} - and (BAB^T) -constraints. This is the point at which the bracket (i) differs from (ii) and (iii).

In the case of bracket (i), because \mathbb{A} and B commute and because the restriction (7.2) is Poisson for \mathbb{A} -algebra, the inter-brackets between \mathbb{A} - and (BAB^T) -constraints vanish as well, so all the above constraints Poisson commute on the total constraint surface, but because they do not commute with all other variables, these constraints are not Poisson and *the Dirac procedure fails*.²

7.4. Dirac procedure for the upper-triangular case. We begin with calculating the constraint matrix. We have two sets of constraints,

$$C_{k,l} = 0 \text{ and } C_{k,l}^* = 0, \text{ for } N \geq k \geq l \geq 0,$$

²In principle, again following Dirac's ideology, one might introduce secondary constraints, but this apparently leads to further reduction of the dimension of the "actual" phase space, which seems to be not feasible.

where

$$(7.4) \quad \begin{aligned} C_{k,l} &= [\mathbb{A}]_{k,l}, \text{ for } N \geq k > l \geq 1, & [\mathbb{A}]_{k,k} - 1, \text{ for } k = 1, \dots, N; \\ C_{k,l}^* &= [B\mathbb{A}B^T]_{k,l} \text{ for } N \geq k > l \geq 1, & [B\mathbb{A}B^T]_{k,k} - 1, \text{ for } k = 1, \dots, N; \end{aligned}$$

the brackets $\{C, C\}$ and $\{C^*, C^*\}$ vanish on the constraint surface $C = C^* = 0$, whereas for the bracket $\{C, C^*\}$ after tedious calculations we obtain the following result (in which we have repeatedly used already the constraint conditions (7.4)):

$$(7.5) \quad \{C_{k,l}, C_{i,j}^*\} = [B]_{i,k} [B\mathbb{A}^T \mathbb{A}]_{j,l} + [B\mathbb{A}\mathbb{A}]_{i,l} [B]_{j,k}.$$

We formulate the condition of the nondegeneracy of this matrix in terms of the corresponding system of linear equations: the constraint matrix (7.5) is nondegenerate iff the matrix equation

$$(7.6) \quad P_{-,1} [B\mathbb{A}\mathbb{A}F B^T + B F^T \mathbb{A}^T \mathbb{A} B^T] = 0$$

where F is a nonstrictly upper-triangular matrix, admits only trivial solutions w.r.t. F . We can immediately simplify this equation observing that, for a general upper-triangular F , the matrix $\mathbb{A}F$ is also upper triangular, so instead of solving Eq. (7.6) we can equivalently solve the equation

$$(7.7) \quad P_{-,1} [B\mathbb{A}F B^T + B F^T \mathbb{A} B^T] = 0$$

again for an upper-triangular F . To simplify this system further, let us introduce the matrix $g = B F^T B^{-1}$. Then, Eq. (7.7) takes the form

$$P_{-,1} [B\mathbb{A} B^T g^T + g B\mathbb{A} B^T] = 0,$$

which was solved by Bondal in [1]: the general solution itself can be written in terms of projection operators (see [3]):

$$(7.8) \quad g = P_{-,1/2} (B\mathbb{A} B^T \omega_-) - P_{+,1/2} (B\mathbb{A}^T B^T \omega_-^T),$$

where ω_- is now *strictly* lower-triangular matrix. Making the substitution $\mathbb{A}' = B\mathbb{A} B^T$ and $B' = B^{-1}$ and using that $F^T = B' g B'^{-1}$ must be lower triangular, we can formulate the condition of nondegeneracy in the following form: given an admissible pair (\mathbb{A}', B') , i.e., such that $B' \mathbb{A}' B'^T$ belongs to the set \mathcal{A} , the equation

$$(7.9) \quad P_+ [B' g (B')^{-1}] = P_+ [B' [P_{-,1/2} (\mathbb{A}' \omega_-) - P_{+,1/2} (\mathbb{A}'^T \omega_-^T)] (B')^{-1}] = 0$$

must have only trivial solutions in the space of strictly lower triangular matrices ω_- .

We first calculate the determinant of the system (7.9) for $B' = \mathbb{E}$. In this case, obviously, this system reads $-P_+ (\mathbb{A}^T \omega_-^T) = 0$, and the determinant is equal to the unity irrespectively on \mathbb{A} . This means that for B close to the unit matrix, this determinant will be nonzero and we can therefore define the Dirac bracket in a layer over the base space \mathcal{A} . This result establishes a link between this approach and Bondal's one: the neighbourhood of the identity defines the groupoid of morphisms which preserve the space of upper-triangular bilinear forms with 1 on the diagonal. The pair $(B\mathbb{A} B^T, g)$ where $g = B F^T B^{-1}$, belongs to the corresponding Lie algebroid.

We now address the problem of choosing a "convenient" parameterization of the (\mathbb{A}, B) -pairs. In the upper-triangular case, we can express all $a_{i,j}$ with $i < j$

through entries of the matrix B . For this, we write the set of conditions implying that the matrix $B\mathbb{A}B^T$ is upper-triangular:

$$(7.10) \quad \sum_{1 \leq i < j \leq n} b_{k,i} b_{l,j} a_{i,j} = - \sum_{s=1}^n b_{k,s} b_{l,s}, \quad n \geq k > l \geq 1.$$

Additionally, we have a restriction due to Bondal [1] on the minors M_d^\pm (see formulas (6.2) and (6.1) of the matrix B : provided $\det B = 1$, $M_d^+ = (-1)^{d(n-d)} M_{n-d}^-$, and we assume that all these minors are nonzero. We have the following technical lemma concerning the determinant of the system (7.10) of linear equations w.r.t. the entries the matrix \mathbb{A} .

Lemma 7.5. *The determinant of the $[n(n-1)/2] \times [n(n-1)/2]$ -matrix $\mathfrak{F}_{i < j}^{k > l} := b_{k,i} b_{l,j}$ is equal to $\prod_{d=1}^{n-1} [M_d^+ M_d^-]$ and is therefore nonzero provided all upper-right and lower-left minors of the matrix B are nonzero.*

The **proof** uses the Bondal's technique of skew-symmetric forms and can be performed by induction in the size of the matrix B .

We can therefore always express $a_{i,j}$ in terms of B writing

$$(7.11) \quad a_{i,j} = F_{i,j}[B].$$

We can also write entries of the transformed matrix $B\mathbb{A}B^T$ in the form

$$(7.12) \quad (B\mathbb{A}B^T)_{i,j} = BF[B]B^T = \tilde{F}[B].$$

Note that the thus defined matrix $\tilde{F}[B]$ is automatically upper-triangular.

We can therefore replace the set of original constraints (7.4) by the equivalent set

$$(7.13) \quad \begin{aligned} a_{i,j} - F_{i,j}[B] &= 0, \quad i < j, & M_d^- &= (-1)^{d(n-d)} M_{n-d}^+, \\ a_{k,l} &= 0, \quad k > l, & a_{k,k} &= 1, \quad k = 1, \dots, n. \end{aligned}$$

We now evaluate the brackets between functions $F_{i,j}[B]$. We begin with case (i) in which the brackets between entries $b_{i,j}$ are just the Lie–Poisson brackets (6.5). In this case, all constraint equations Poisson commute and \mathbb{A} commutes with B , so we have

$$(7.14) \quad \{a_{i,j} - F_{i,j}[B], a_{s,p} - F_{s,p}[B]\} = \{a_{i,j}, a_{s,p}\} + \{F_{i,j}[B], F_{s,p}[B]\} = 0,$$

or

$$(7.15) \quad \{F_{i,j}[B], F_{s,p}[B]\} = -\{a_{i,j}, a_{s,p}\}|_{\mathbb{A}=F[B]}$$

(note the minus sign in this relation). The brackets between entries of F are here induced by the standard Lie–Poisson bracket (1.5).

In the case (ii) or (iii), we use the Dirac procedure that implies that brackets between constraints as well as brackets between constraints and all variables vanish on the constraint surface. For instance,

$$\{a_{i,j}, a_{s,p} - F_{s,p}[B]\}_D = 0, \text{ i.e., } \{a_{i,j}, a_{s,p}\}_D = \{a_{i,j}, F_{s,p}[B]\}_D,$$

or, since the $\{a, a\}$ brackets are not changed by the Dirac reduction, we obtain that

$$(7.16) \quad \{a_{i,j}, F_{s,p}[B]\}_D|_{\mathbb{A}=F[B]} = \{a_{i,j}, a_{s,p}\}|_{\mathbb{A}=F[B]}.$$

Then for the brackets between entries of the matrix F we obtain

$$\{a_{i,j} - F_{i,j}[B], a_{s,p} - F_{s,p}[B]\}_D = \{a_{i,j}, a_{s,p}\}|_{\mathbb{A}=F[B]}$$

$$-2\{a_{i,j}, a_{s,p}\}|_{\mathbb{A}=F[B]} + \{F_{i,j}[B], F_{s,p}[B]\}_D$$

and therefore

$$(7.17) \quad \{F_{i,j}[B], F_{s,p}[B]\}_D = \{a_{i,j}, a_{s,p}\}|_{\mathbb{A}=F[B]} \text{ for } i < j \text{ and } s < p$$

with the plus sign.

We can also introduce an analogous representation for the matrix $B\mathbb{A}B^T$:

$$(7.18) \quad [B\mathbb{A}B^T]_{i,j} := \begin{cases} \tilde{F}[B]_{i,j}, & i < j; 1, & i = j; 0, & i > j \end{cases} := \tilde{F}[B]$$

Note that if we impose the standard Lie–Poisson brackets on B then (see the proof and discussion in Sec. 10) we obtain that Poisson relations between \tilde{F} are the same as for \mathbb{A} (with the plus sign) and F Poisson commute with \tilde{F} .

If we again consider the case (ii) or (iii) and apply the Dirac procedure, then, since the above reasonings remain valid for the set of reparameterized constraints $[B\mathbb{A}B^T]_{i,j} - \tilde{F}[B]_{i,j} = 0$ and the Dirac brackets between entries of $B\mathbb{A}B^T$ coincide with the initial brackets, we obtain that the brackets between entries of \tilde{F} are

$$(7.19) \quad \{\tilde{F}_{i,j}[B], \tilde{F}_{s,p}[B]\}_D = \{a_{i,j}, a_{s,p}\}|_{\mathbb{A}=\tilde{F}[B]}.$$

We can take the set $\{F_{i,j}, \tilde{F}_{s,p}\}$ as *new dynamical variables* describing our system; these variables are not however algebraically independent as they share $[n/2]$ Casimir functions Y_p generated by $\det(A + \lambda A^T)$. But, say, in the case (ii) we also have $[n/2]$ additional Casimir functions X_p (see Theorem 6.3). Note that for an upper-triangular matrix \mathbb{A} the principal minors of the matrix $\mathbb{A}^T B^{-1}$ coincide with those of the matrix B^{-1} , so all X_p can be expressed as ratios of principal and upper-right minors of B and are algebraically independent. We can therefore parameterize the general Poisson leaf of the Dirac Poisson algebra by the variables $F_{i,j}$ and $\tilde{F}_{s,p}$; inside each of these two sets the brackets are given by those of the entries of \mathbb{A} , the values of the Casimir functions Y_p coincide for these two sets, and the brackets between F and \tilde{F} are determined by the Dirac procedure; we do not evaluate these brackets in this paper and only mention that they are nontrivial.

8. QUANTIZATION

In this section, we quantize the new algebras (1.6) for nontrivial Q . As above, we concentrate mostly on the case (ii).

We use the standard trigonometric quantum R -matrix

$$(8.1) \quad R_{12}(q) = \mathbb{E} \otimes \mathbb{E} + \sum_{k,l} \overset{1}{E}_{k,l} \otimes \overset{2}{E}_{l,k} \left[(q - q^{-1})\theta(l - k) + \frac{(q^{1/2} - q^{-1/2})^2}{2} \delta_{k,l} \right].$$

The matrix $R_{12}(q)$ manifests the following useful properties:

$$(8.2) \quad [R_{12}(q)]^{-1} = R_{12}(q^{-1})$$

$$(8.3) \quad [R_{12}(q), R_{12}^{t_1}(q)] = [R_{12}(q), R_{12}^{t_2}(q)] = 0,$$

$$(8.4) \quad R_{12}(q) + R_{21}(q^{-1}) = (q - q^{-1})P_{12}$$

$$(8.5) \quad R_{12}(q)R_{13}(q)R_{23}(q) = R_{23}(q)R_{13}(q)R_{12}(q) \quad \text{the Yang–Baxter relation.}$$

We use the standard notation: the entries of the matrices $\mathbb{A} = E_{i,j} \otimes a_{i,j}$ and $B = E_{i,j} \otimes b_{i,j}$ are now operators in the quantum space. The orders of multiplication in the classical space and in the quantum space can be in principle different.

However, when not stated explicitly, we assume that the order of multiplication in the quantum space is *natural*, i.e., it coincides with the ordering of the matrices \mathbb{A} and B in matrix products.

Lemma 8.1. *The quantum version of the case (ii) Poisson algebra reads:*

$$(8.6) \quad R_{12}(q) \overset{1}{\mathbb{A}} R_{12}^{t_1}(q) \overset{2}{\mathbb{A}} = \overset{2}{\mathbb{A}} R_{12}^{t_1}(q) \overset{1}{\mathbb{A}} R_{12}(q)$$

$$(8.7) \quad R_{12}(q) \overset{1}{B} \overset{2}{B} = \overset{2}{B} \overset{1}{B} R_{12}(q)$$

$$(8.8) \quad \overset{2}{\mathbb{A}} \overset{1}{B} R_{12}(q) = \overset{1}{B} R_{12}^{t_2}(q^{-1}) \overset{2}{\mathbb{A}}$$

These commutation relations satisfy the Jacobi relations and ensure the quantum automorphism: the products $B\mathbb{A}B^T$ satisfy the quantum algebra (8.6).

The *proof* is a straightforward but lengthy calculation alongside which we encounter another Yang–Baxter relation,

$$(8.9) \quad R_{23}(q) R_{13}^{t_1}(q^{-1}) R_{12}^{t_1}(q^{-1}) = R_{12}^{t_1}(q^{-1}) R_{13}^{t_1}(q^{-1}) R_{23}(q),$$

which can be derived from the original relation (8.5) by total transposition in the first space (note that under this operation entries with R_{13} and R_{12} permute and the entry with R_{23} retains its position). After that, sandwiching the both sides of the obtained relation between two insertions of $R_{12}(q^{-1})$, we obtain the original Yang–Baxter relation with the global replacement $q \rightarrow q^{-1}$.

Note that it is safe to transpose commutation relations provided we *do not* change the order of multiplication in the quantum space (and entries in the quantum space commute with all R -matrix entries). For instance, from (8.7) we have that

$$(8.10) \quad B^T R_{12}^{t_1}(q) \overset{2}{B} = \overset{2}{B} R_{12}^{t_1}(q) B^T$$

and

$$(8.11) \quad B^T \overset{1}{B} \overset{2}{B} R_{12}^{t_1 t_2}(q) = R_{12}^{t_1 t_2}(q) B^T \overset{2}{B} \overset{1}{B}$$

and since $R_{12}(q) + R_{21}(q^{-1}) = P_{12}$ we can first replace $R_{12}^{t_1 t_2}(q) = R_{21}(q)$ by $-R_{12}(q^{-1})$ and then, multiplying the relation (8.11) by $R_{12}(q)$ from both sides, we obtain that

$$(8.12) \quad R_{12}(q) B^T \overset{1}{B} \overset{2}{B} = \overset{2}{B} \overset{1}{B} B^T R_{12}(q).$$

From (8.8) we have that

$$(8.13) \quad \overset{2}{\mathbb{A}} R_{12}^{t_1}(q) B^T = R_{12}^{t_1 t_2}(q^{-1}) B^T \overset{1}{\mathbb{A}}.$$

The proof of the second statement of the lemma (the automorphism) follows from the following chain of equalities:

$$\begin{aligned} & R_{12}(q) \overset{1}{B} \overset{1}{\mathbb{A}} [B^T R_{12}^{t_1}(q) \overset{2}{B}] \overset{2}{\mathbb{A}} B^T = R_{12}(q) \overset{1}{B} \overset{1}{\mathbb{A}} \overset{2}{B} R_{12}^{t_1}(q) B^T \overset{1}{\mathbb{A}} B^T \\ &= R_{12}(q) \overset{1}{B} [\overset{1}{\mathbb{A}} \overset{2}{B} R_{21}(q)] R_{12}^{t_1}(q) [R_{21}(q^{-1}) B^T \overset{1}{\mathbb{A}}] B^T \\ &= [R_{12}(q) \overset{1}{B} \overset{2}{B}] R_{12}^{t_2}(q^{-1}) \overset{1}{\mathbb{A}} R_{12}^{t_1}(q) \overset{2}{\mathbb{A}} R_{12}^{t_1}(q) B^T \overset{1}{\mathbb{A}} B^T \\ &= \overset{2}{B} \overset{1}{B} [R_{12}(q) R_{12}^{t_2}(q^{-1})] \overset{1}{\mathbb{A}} R_{12}^{t_1}(q) \overset{2}{\mathbb{A}} R_{12}^{t_1}(q) B^T \overset{1}{\mathbb{A}} B^T \end{aligned}$$

$$\begin{aligned}
&= {}^2_1 B {}^1 R_{12}^{t_2}(q^{-1}) [R_{12}(q) {}^1 \mathbb{A} {}^2 R_{12}^{t_1}(q) {}^2 \mathbb{A}] R_{12}^{t_1}(q) B^T B^T \\
&= {}^2_1 B {}^1 R_{12}^{t_2}(q^{-1}) {}^2 \mathbb{A} R_{12}^{t_1}(q) {}^1 \mathbb{A} [R_{12}(q) R_{12}^{t_1}(q)] B^T B^T \\
&= {}^2_1 B {}^1 R_{12}^{t_2}(q^{-1}) {}^2 \mathbb{A} R_{12}^{t_1}(q) {}^1 \mathbb{A} R_{12}^{t_1}(q) [R_{12}(q) B^T B^T] \\
&= {}^2_1 B [{}^1 B R_{12}^{t_2}(q^{-1}) {}^2 \mathbb{A}] R_{12}^{t_1}(q) [{}^1 \mathbb{A} R_{12}^{t_1}(q) B^T] B^T R_{12}(q) \\
&= {}^2_2 B {}^1 \mathbb{A} [R_{12}(q) R_{12}^{t_1}(q) R_{12}(q^{-1})] B^T {}^1 \mathbb{A} B^T R_{12}(q) \\
&= {}^2_2 B [{}^1 B R_{12}^{t_1}(q) B^T] {}^1 \mathbb{A} B^T R_{12}(q) = {}^2_2 B {}^2 \mathbb{A} B^T R_{12}^{t_1}(q) {}^1 B {}^1 \mathbb{A} B^T R_{12}(q).
\end{aligned}$$

We prove the Jacobi relations for the $\{\mathbb{A}, \mathbb{A}, B\}$ -commutator, the proof for the $\{\mathbb{A}, B, B\}$ -commutator is analogous. Making the cyclic change of indices $(1, 2, 3) \rightarrow (2, 3, 1)$ and doing the total transposition of (8.9) we obtain that

$$(8.14) \quad R_{23}^{t_3}(q^{-1}) R_{12}^{t_2}(q^{-1}) R_{13}(q) = R_{13}(q) R_{12}^{t_2}(q^{-1}) R_{23}^{t_3}(q^{-1}).$$

Using that $R_{23}^{t_3}(q^{-1}) + R_{32}^{t_3}(q) = (q - q^{-1})P_{23}^{t_3}$ and that

$$P_{23}^{t_3} R_{12}^{t_2}(q^{-1}) R_{13}(q) = P_{23}^{t_3} R_{13}(q^{-1}) R_{13}(q) = P_{23}^{t_3}$$

and

$$R_{13}(q) R_{12}^{t_2}(q^{-1}) P_{23}^{t_3} = R_{13}(q) R_{13}(q^{-1}) P_{23}^{t_3} = P_{23}^{t_3},$$

we can effectively replace $R_{23}^{t_3}(q^{-1})$ by $R_{32}^{t_3}(q) = R_{23}^{t_2}(q)$ in (8.14) thus obtaining another Yang–Baxter-type relation

$$(8.15) \quad R_{23}^{t_2}(q) R_{12}^{t_2}(q^{-1}) R_{13}(q) = R_{13}(q) R_{12}^{t_2}(q^{-1}) R_{23}^{t_2}(q).$$

The Jacobi relation now follows from the chain of equalities

$$\begin{aligned}
& {}^3 \mathbb{A} R_{23}^{t_2}(q) [{}^2_1 \mathbb{A} B R_{12}(q)] R_{13}(q) R_{23}(q) \\
&= {}^3_1 \mathbb{A} B [R_{23}^{t_2}(q) R_{12}^{t_2}(q^{-1}) R_{13}(q)] {}^2 \mathbb{A} R_{23}(q) \\
&= [{}^3_1 \mathbb{A} B R_{13}(q)] R_{12}^{t_2}(q^{-1}) R_{23}^{t_2}(q) {}^2 \mathbb{A} R_{23}(q) \\
&= [{}^3_1 \mathbb{A} B R_{13}(q)] R_{12}^{t_2}(q^{-1}) R_{23}^{t_2}(q) {}^2 \mathbb{A} R_{23}(q) \\
&= {}^1 B R_{13}^{t_3}(q^{-1}) R_{12}^{t_2}(q^{-1}) [{}^3 \mathbb{A} R_{23}^{t_2}(q) {}^2 \mathbb{A} R_{23}(q)] \\
&= {}^1 B [R_{13}^{t_3}(q^{-1}) R_{12}^{t_2}(q^{-1}) R_{23}(q)] {}^2 \mathbb{A} R_{23}^{t_2}(q) {}^3 \mathbb{A} \\
&= R_{23}(q) [{}^1 B R_{12}^{t_2}(q^{-1}) {}^2 \mathbb{A}] R_{13}^{t_3}(q^{-1}) R_{23}^{t_2}(q) {}^3 \mathbb{A} \\
&= R_{23}(q) {}^2_1 \mathbb{A} B [R_{12}(q) R_{13}^{t_3}(q^{-1}) R_{23}^{t_2}(q)] {}^3 \mathbb{A} \\
&= R_{23}(q) {}^2 \mathbb{A} R_{23}^{t_2}(q) [{}^1 B R_{13}^{t_3}(q^{-1}) {}^3 \mathbb{A}] R_{12}(q) \\
&= [R_{23}(q) {}^2 \mathbb{A} R_{23}^{t_2}(q) {}^3 \mathbb{A}] {}^1 B R_{13}(q) R_{12}(q) \\
&= {}^3 \mathbb{A} R_{23}^{t_2}(q) {}^2_1 \mathbb{A} B R_{23}(q) R_{13}(q) R_{12}(q),
\end{aligned}$$

and using the standard Yang–Baxter relation (8.5) we obtain the identity. The lemma is proved.

9. QUANTUM AFFINE ALGEBRAS

Similarly to the approach of paper [3] we can introduce the affine version of the quantum relations. We now have infinite-dimensional quantum algebras with the generators

$$(9.1) \quad \mathbb{A}(\lambda) = \sum_{i,j} \sum_{k=0}^{\infty} E_{i,j} \otimes a_{i,j}^{(k)} \lambda^{-k}, \quad B(\lambda) = \sum_{i,j} \sum_{k=0}^{\infty} E_{i,j} \otimes b_{i,j}^{(k)} \lambda^{-k}.$$

We also have the quantum affine R -matrix (depending on the spectral parameters λ and μ):

$$(9.2) \quad R_{12}(\lambda, \mu; q) := \frac{\lambda - \mu}{q^{-1}\lambda - q\mu} \sum_{i \neq j} \overset{1}{E}_{i,i} \otimes \overset{2}{E}_{j,j} + \sum_i \overset{1}{E}_{i,i} \otimes \overset{2}{E}_{i,i} \\ + \frac{(q^{-1} - q)\lambda}{q^{-1}\lambda - q\mu} \sum_{i < j} \overset{1}{E}_{i,j} \otimes \overset{2}{E}_{j,i} + \frac{(q^{-1} - q)\mu}{q^{-1}\lambda - q\mu} \sum_{i > j} \overset{1}{E}_{i,j} \otimes \overset{2}{E}_{j,i}.$$

We choose the normalization such that

$$(9.3) \quad [R_{12}(\lambda, \mu; q)]^{-1} = R_{12}(\lambda, \mu; q^{-1}).$$

We also have several useful relations:

$$(9.4) \quad R_{12}(\lambda, \mu; q) = R_{12}^{t_1 t_2}(\lambda^{-1}, \mu^{-1}; q^{-1}),$$

$$(9.5) \quad R_{12}(\lambda, \mu; q) = R_{12}(\mu^{-1}, \lambda^{-1}; q),$$

$$(9.6) \quad R_{12}(\lambda, \mu; q) R_{12}^{t_1}(\rho, \nu; q^{-1}) R_{12}(\lambda, \mu; q^{-1}) = R_{12}^{t_1}(\rho, \nu; q^{-1}).$$

The R -matrix $R_{12}(\lambda, \mu; q)$ satisfies the Yang–Baxter relations

$$(9.7) \quad R_{12}(\lambda, \mu; q) R_{13}(\lambda, \rho; q) R_{23}(\mu, \rho; q) = R_{23}(\mu, \rho; q) R_{13}(\lambda, \rho; q) R_{12}(\lambda, \mu; q)$$

The main lemma reads

Lemma 9.1. *The affine quantum version of the case (ii) Poisson algebra reads:*

$$(9.8) \quad R_{12}(\lambda, \mu; q) \overset{1}{\mathbb{A}}(\lambda) R_{12}^{t_1}(\lambda^{-1}, \mu; q) \overset{2}{\mathbb{A}}(\mu) = \overset{2}{\mathbb{A}}(\mu) R_{12}^{t_1}(\lambda^{-1}, \mu; q) \overset{1}{\mathbb{A}}(\lambda) R_{12}(\lambda, \mu; q)$$

$$(9.9) \quad R_{12}(\lambda, \mu; q) \overset{1}{B}(\lambda) \overset{2}{B}(\mu) = \overset{2}{B}(\mu) \overset{1}{B}(\lambda) R_{12}(\lambda, \mu; q)$$

$$(9.10) \quad \overset{2}{\mathbb{A}}(\mu) \overset{1}{B}(\lambda) R_{12}(\lambda, \mu; q) = \overset{1}{B}(\lambda) R_{12}^{t_2}(\mu, \lambda^{-1}; q^{-1}) \overset{2}{\mathbb{A}}(\mu)$$

Note that in the last relation we can equivalently substitute

$$R_{12}^{t_2}(\mu, \lambda^{-1}; q^{-1}) = R_{12}^{t_2}(\lambda, \mu^{-1}; q^{-1}).$$

These commutation relations satisfy the Jacobi relations and ensure the quantum automorphism: the products

$$B(\lambda) \mathbb{A}(\lambda) B^T(\lambda^{-1})$$

satisfy the quantum algebra (9.8).

The proof is straightforward. We have a Yang–Baxter-type relation

$$(9.11) \quad R_{23}(\mu, \nu; q) R_{12}^{t_2}(\lambda, \mu^{-1}; q^{-1}) R_{13}^{t_3}(\lambda, \nu^{-1}; q^{-1}) \\ = R_{13}^{t_3}(\lambda, \nu^{-1}; q^{-1}) R_{12}^{t_2}(\lambda, \mu^{-1}; q^{-1}) R_{23}(\mu, \nu; q),$$

which can again be derived from the standard relation (9.7) if we perform the transposition in the first space (then, again, items with R_{13} and R_{12} permute)

and take relation (9.4) into account. After this, the two chains of calculations are completely analogous to those in the proof of Lemma 8.1; we just need to set the spectral parameters in the proper places and use the corresponding identities. For example, we obtain the commutation relations between $B^T(\lambda^{-1})$ and $B^T(\mu^{-1})$ by transposing relations (9.9) and substituting $(\lambda, \mu) \rightarrow (\lambda^{-1}, \mu^{-1})$, which gives

$$B^T(\lambda^{-1})B^T(\mu^{-1})R_{12}^{t_1 t_2}(\lambda^{-1}, \mu^{-1}; q) = R_{12}^{t_1 t_2}(\lambda^{-1}, \mu^{-1}; q)B^T(\mu^{-1})B^T(\lambda^{-1}).$$

But

$$R_{12}^{t_1 t_2}(\lambda^{-1}, \mu^{-1}; q) = R_{12}(\lambda, \mu; q^{-1}) = [R_{12}(\lambda, \mu; q)]^{-1},$$

so multiplying from the both sides by $R_{12}(\lambda, \mu; q)$, we obtain the familiar relation

$$R_{12}(\lambda, \mu; q)B^T(\lambda^{-1})B^T(\mu^{-1}) = B^T(\mu^{-1})B^T(\lambda^{-1})R_{12}(\lambda, \mu; q).$$

We omit the proof of the algebra automorphism as this proof repeats that in Lemma 8.1; we only present the proof of the Jacobi relations for the $\{\mathbb{A}(\lambda), B(\mu), B(\nu)\}$ -triple relations:

$$\begin{aligned} & \mathbb{A}(\lambda) [\overset{3}{B}(\nu) \overset{2}{B}(\mu) R_{23}(\mu, \nu; q)] R_{21}(\mu, \lambda; q) R_{31}(\nu, \lambda; q) \\ &= R_{23}(\mu, \nu; q) [\mathbb{A}(\lambda) \overset{2}{B}(\mu) R_{21}(\mu, \lambda; q)] \overset{3}{B}(\nu) R_{31}(\nu, \lambda; q) \\ &= R_{23}(\mu, \nu; q) \overset{2}{B}(\mu) R_{12}^{t_2}(\lambda, \mu^{-1}; q^{-1}) [\mathbb{A}(\lambda) \overset{3}{B}(\nu) R_{31}(\nu, \lambda; q)] \\ &= [R_{23}(\mu, \nu; q) \overset{2}{B}(\mu) \overset{3}{B}(\nu)] R_{12}^{t_2}(\lambda, \mu^{-1}; q^{-1}) R_{13}^{t_3}(\lambda, \nu^{-1}; q^{-1}) \mathbb{A}(\lambda) \\ &= \overset{3}{B}(\nu) \overset{2}{B}(\mu) [R_{23}(\mu, \nu; q) R_{12}^{t_2}(\lambda, \mu^{-1}; q^{-1}) R_{13}^{t_3}(\lambda, \nu^{-1}; q^{-1})] \mathbb{A}(\lambda) \\ &= \overset{3}{B}(\nu) R_{13}^{t_3}(\lambda, \nu^{-1}; q^{-1}) [\overset{2}{B}(\mu) R_{12}^{t_2}(\lambda, \mu^{-1}; q^{-1}) \mathbb{A}(\lambda)] R_{23}(\mu, \nu; q) \\ &= [\overset{3}{B}(\nu) R_{13}^{t_3}(\lambda, \nu^{-1}; q^{-1}) \mathbb{A}(\lambda)] \overset{2}{B}(\mu) R_{21}(\mu, \lambda; q) R_{23}(\mu, \nu; q) \\ &= \mathbb{A}(\lambda) \overset{3}{B}(\nu) \overset{2}{B}(\mu) R_{31}(\nu, \lambda; q) R_{21}(\mu, \lambda; q) R_{23}(\mu, \nu; q), \end{aligned}$$

and we again obtain the standard Yang–Baxter relation in the first and last terms. The Jacobi relation is therefore proved.

10. POISSON GROUPOID STRUCTURE

In this section, we investigate the possibility of inducing brackets on the upper-triangular matrix \mathbb{A} from the Lie–Poisson brackets (2.3) on B . We shall demonstrate that the obtained brackets then satisfy the definition of a Poisson (symplectic) groupoid [23], [16].

As we have already demonstrated in Sec. 7, we can express entries of \mathbb{A} as the functions $F_{i,j}[B]$; we have already shown that the brackets between $F_{i,j}[B]$ and $F_{s,p}[B]$ are given by minus the bracket between $a_{i,j}$ and $a_{s,p}$ upon the substitution $\mathbb{A} = F[B]$. We now evaluate all other brackets of this system.

We begin with evaluating the bracket between $F_{i,j}[B]$ and $b_{s,p}$ induced by the bracket (2.3). Note first that

$$\begin{aligned} \{b_{k,i} b_{l,j}, b_{s,p}\} &= \theta(i-p)[b_{k,p} b_{l,j}] b_{s,i} - \theta(s-k)[b_{s,i} b_{l,j}] b_{k,p} \\ &\quad - \theta(p-j)[b_{k,i} b_{l,p}] b_{a,j} + \theta(l-s)[b_{k,i} b_{s,j}] b_{l,p}. \end{aligned} \tag{10.1}$$

In this expression we group in the square brackets in the r.h.s. the terms that are entries of the matrix $\mathfrak{F}_{\alpha < \beta}^{\kappa > \lambda} := b_{\kappa, \alpha} b_{\lambda, \beta}$ provided the original combination $b_{k, i} b_{l, j}$ was an entry of this matrix. If we take $i = j$ in (10.1) then in the r.h.s. we have either terms with coinciding right indices or terms that are again entries of \mathfrak{F} . We are therefore able to evaluate explicitly the Poisson bracket of $b_{s, p}$ with

$$F_{i, j}[B] = \sum_{k > l} [\mathfrak{F}^{-1}]_{i < j}^{k > l} \left[\sum_s b_{k, s} b_{l, s} \right].$$

After a tedious algebra we obtain a compact answer, which can be written in a convenient r -matrix form: taking $\mathbb{F} := \{F_{i, j}[B], i < j; 1, i = j, 0, i > j\}$ to be an upper-triangular matrix, we merely obtain that

$$(10.2) \quad \{B \otimes \mathbb{F}\} = \frac{1}{B} r_{12} \mathbb{F} - \frac{1}{B} \mathbb{F} r_{12}^t,$$

or, in the component form,

$$(10.3) \quad \{b_{i, j}, f_{k, l}\} = \sum_s \theta(s - j) \delta_{k, j} b_{i, s} f_{s, l} - \sum_s \theta(j - s) \delta_{j, l} b_{i, s} f_{k, s}.$$

We can take the above formula as the *definition* of a new bracket between B and \mathbb{F} assuming that \mathbb{F} have the Poisson bracket of the form (1.2) *with the overall negative sign*,

$$(10.4) \quad \{\mathbb{F} \otimes \mathbb{F}\} = -r_{12}(\mathbb{F} \mathbb{F}) + (\mathbb{F} \mathbb{F}) r_{12} - \mathbb{F} r_{12}^t \mathbb{F} + \mathbb{F} r_{12}^t \mathbb{F},$$

and B have the standard brackets (2.3). If we then perform the mapping $\mathbb{F} \mapsto \tilde{\mathbb{F}} := B \mathbb{F} B^T$ we obtain using only (2.3), (10.2), and (10.4) the following Poisson relations:

$$(10.5) \quad \{\tilde{\mathbb{F}} \otimes \tilde{\mathbb{F}}\} = r_{12}(\tilde{\mathbb{F}} \tilde{\mathbb{F}}) - (\tilde{\mathbb{F}} \tilde{\mathbb{F}}) r_{12} + \tilde{\mathbb{F}} r_{12}^t \tilde{\mathbb{F}} - \tilde{\mathbb{F}} r_{12}^t \tilde{\mathbb{F}},$$

$$(10.6) \quad \{B \otimes \tilde{\mathbb{F}}\} = r_{12}(B \tilde{\mathbb{F}}) - \tilde{\mathbb{F}} r_{12}^t B.$$

The component form of writing of the latter relation is

$$(10.7) \quad \{b_{i, j}, \tilde{f}_{k, l}\} = \theta(i - k) b_{k, j} \tilde{f}_{i, l} - \theta(l - i) b_{l, j} \tilde{f}_{k, i}.$$

Note first that this new Poisson algebra of \mathbb{F} and B satisfies all the Jacobi relations even if we consider *independent* pairs $(\mathbb{F}, B) \in GL_N \times GL_N$. Actually, the most economic way to see the satisfaction of the Poisson Jacobi relations is by *quantizing* the corresponding algebra; the corresponding *quantum commutation relations* are

$$(10.8) \quad \tilde{\mathbb{F}} R_{12}^{t_1}[q] B = R_{12}[q] B \tilde{\mathbb{F}}.$$

Second, this algebra admits Poisson reduction to any block-upper triangular form: all the constraints $\mathbb{F}_{I, J} = 0$ for $I > J$ and $\tilde{\mathbb{F}}_{I, J} = 0$ for $I > J$ are now Poissonian. Third, evaluating the brackets between \mathbb{F} and $\tilde{\mathbb{F}}$ we merely obtain that

$$\{\mathbb{F} \otimes \tilde{\mathbb{F}}\} = 0,$$

so these algebras are in fact totally separated.

These two conditions: that $\mathbb{F} \rightarrow \tilde{\mathbb{F}} = B \mathbb{F} B^T$ is an anti-Poisson mapping and that \mathbb{F} Poisson commutes with $\tilde{\mathbb{F}}$ imply that these Poisson brackets are in fact those for

a symplectic groupoid [23]. We can therefore formulate the following lemma and conjecture.

Lemma 10.1. *Considering a pair $(\mathbb{F}, B) \in GL_N \times GL_N$ endowed with the brackets (2.3), (10.2), and (10.4) we have that*

- (i) *the source $s : (\mathbb{F}, B) \rightarrow \mathbb{F}$ and target $t : (\mathbb{F}, B) \rightarrow \tilde{\mathbb{F}} \equiv B\mathbb{F}B^T$ mappings are correspondingly antiautomorphism and automorphism of the Poisson algebra (1.2);*
- (ii) *the above algebraic elements \mathbb{F} and $\tilde{\mathbb{F}}$ Poisson commute;*
- (iii) *restrictions of \mathbb{F} and $\tilde{\mathbb{F}}$ to any b.u.t. form are Poissonian w.r.t. the total Poisson algebra of the pair (\mathbb{F}, B) ;*
- (iv) *in the upper-triangular case, the Poisson algebra for $\mathbb{F} = \mathbb{A}$ is induced by the Lie–Poisson brackets (1.5) for B ;*
- (v) *the quantum version of the Poisson (\mathbb{F}, B) -algebra is given by (8.7), (10.8), and the relation*

$$(10.9) \quad R_{12}(q^{-1})^1 \mathbb{F} R_{12}^{t_1}(q^{-1})^2 \mathbb{F} = {}^2 \mathbb{F} R_{12}^{t_1}(q^{-1})^1 \mathbb{F} R_{12}(q^{-1})$$

inverse to (8.6).

We also observe that the above brackets satisfy the definition of the Poisson (symplectic) groupoid (see [23], [16])

Notation 10.2. A *Poisson (symplectic) groupoid* Γ is a symplectic manifold (Γ, Ω) , where Ω is a Poisson structure such that the graph $\mathcal{M} := \{(x, y, m(x, y)) \in \Gamma \times \Gamma \times \Gamma | (x, y) \in \Gamma_2\}$ of the groupoid multiplication m is a Lagrangian submanifold of

$$(\Gamma, \Omega) \times (\Gamma, \Omega) \times (\Gamma, -\Omega).$$

In other words, given three mutually Poisson commuting pairs $(\mathbb{F}_i, B_i) \in GL_N \times GL_N$, $i = 1, 2, 3$ each endowed with a Poisson structure $\pm\Omega$ with the plus signs for the first two pairs and the minus sign for the third pair, we obtain the structure of the Poisson groupoid if the following three sets of constraints are Lagrangian:

$$(10.10) \quad f := B_3 - B_2 B_1, \quad g := \mathbb{F}_3 - \mathbb{F}_1, \quad h := \mathbb{F}_2 - B_1 \mathbb{F}_1 B_1^T,$$

i.e., all these constraints Poisson commute on the constraint surface.

Lemma 10.3. *The pair $\Gamma := (\mathbb{F}, B)$ endowed with the Poisson bracket Ω defined by the Poisson relations (2.3), (10.2), and (10.4) satisfies Definition 10.2 of the Poisson (symplectic) groupoid.*

The *proof* is the direct calculation: taking three mutually commuting pairs (\mathbb{F}_i, B_i) endowed with the respective Poisson, Poisson, and anti-Poisson structures, we can easily verify that all six brackets $\{f, f\}$, $\{g, g\}$, $\{h, h\}$, $\{f, g\}$, $\{f, h\}$, $\{g, h\}$ vanish on the constraint surface. Note that in order for the last bracket $\{g, h\} = \{\mathbb{F}_1, \tilde{\mathbb{F}}_1\}$ to vanish, the source and target projections of the pair (\mathbb{F}, B) must Poisson commute.

Conjecture 10.4. Lemma 10.3 implies the existence of the symplectic form on the pairs (\mathbb{F}, B) for the b.u.t.-restricted matrices \mathbb{F} . This symplectic structure must generalize the Bondal's symplectic structure [1] on the pairs (\mathbb{A}, B) in the upper-triangular case.

Note that the mapping $\mathbb{F} \mapsto B\mathbb{F}B^T$ is now an antiautomorphism of the Poisson algebra for \mathbb{F} . This complies with the Poisson groupoid construction but is opposite to the ideology of Poisson symmetric spaces, which we advocate in the main part of the paper. One reason for us to “dislike” an otherwise nice symplectic groupoid construction is that because of the total separation of variables \mathbb{F} and $\tilde{\mathbb{F}}$, the dynamics described by the mapping $\mathbb{F} \mapsto B\mathbb{F}B^T$ becomes trivial.³

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APPENDIX A NOTATION

In this paper we use the standard notation

$$\overset{1}{\mathbb{A}} = \mathbb{A} \otimes E, \quad \text{and} \quad \overset{2}{\mathbb{A}} = E \otimes \mathbb{A},$$

where E is the $N \times N$ identity matrix. The expression $r_{12}^{t_i}$ denotes the matrix (1.3) transposed w.r.t. the arguments of the i th space ($i = 1, 2$). The r -matrix satisfies the classical Yang–Baxter equation ensuring the Jacobi identities for the Poisson brackets and the additional property that

$$(A.1) \quad r_{12} + r_{12}^{t_1 t_2} = 2P_{12},$$

where $P_{12} = \sum_{i,j} \overset{1}{E}_{i,j} \otimes \overset{2}{E}_{j,i}$ is the standard permutation r -matrix satisfying

$$(A.2) \quad P_{12}(\overset{1}{A} \otimes \overset{2}{B}) = (\overset{1}{B} \otimes \overset{2}{A})P_{12}$$

for any matrices A and B .

In this notation, $\{\overset{1}{\mathbb{A}} \otimes \overset{2}{\mathbb{A}}\}$ is a tensor of 4 components. To extract the bracket between entries $a_{i,j}$ and $a_{k,l}$ of the matrix \mathbb{A} we need to compute the ik_{jl} component of the tensor $r_{12}(\overset{1}{\mathbb{A}} \otimes \overset{2}{\mathbb{A}}) - (\overset{1}{\mathbb{A}} \otimes \overset{2}{\mathbb{A}})r_{12} + \overset{1}{\mathbb{A}}r_{12}^{t_1} \overset{2}{\mathbb{A}} - \overset{2}{\mathbb{A}}r_{12}^{t_1} \overset{1}{\mathbb{A}}$. This gives (1.1).

³A minor subtlety appear when concerning central elements of this algebra: \mathbb{F} and $\tilde{\mathbb{F}}$ share the same set of elements Y_p that are central for both these sets; however the original Poisson–Lie bracket for B has the full Poisson dimension $n(n-1)$, so we must have additional $[n/2]$ elements Q_p that are, first, algebraically independent with \mathbb{F} and $\tilde{\mathbb{F}}$ and, second, do not commute with Y_p . The constraints $Y_p = c_p$, $Q_p = 0$, $p = 1, \dots, [n/2]$, are then of the second kind, the matrix $\{Y_p, Q_p\}$ is nondegenerate, and we can again implement the Dirac procedure w.r.t. these constraints; this procedure does not change the brackets inside the set $\{\mathbb{F}, \tilde{\mathbb{F}}\}$ of remaining dynamical variables.

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